UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 9:00 am to 12:00 pm

PAPER 41

QUANTUM FIELD THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The free Klein–Gordon field obeys the equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \,.$$

Using Noether's theorem find the expressions for the conserved energy and conserved three-momentum $\mathbf{P}$.

In the quantised theory the field $\phi(\mathbf{x})$ and the conjugate field $\pi(\mathbf{x})$ can be expressed as

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(a_\mathbf{p} e^{i\mathbf{p}.\mathbf{x}} + a^\dagger_\mathbf{p} e^{-i\mathbf{p}.\mathbf{x}} \right)$$
$$\pi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_\mathbf{p}}{2}} \left(a_\mathbf{p} e^{i\mathbf{p}.\mathbf{x}} - a^\dagger_\mathbf{p} e^{-i\mathbf{p}.\mathbf{x}} \right) \,.$$

in the Schrödinger picture, where $E_\mathbf{p} = (\mathbf{p}^2 + m^2)^{1/2}$. Write down the commutation relations satisfied by $a_\mathbf{p}$ and $a^\dagger_\mathbf{p}$.

Show that the Hamiltonian and $\mathbf{P}$ can be expressed as

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_\mathbf{p}\, a^\dagger_\mathbf{p}\, a_\mathbf{p} \,,$$
$$\mathbf{P} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p}\, a^\dagger_\mathbf{p}\, a_\mathbf{p} \,,$$

in the quantised theory.

Calculate $[H, a^\dagger_\mathbf{p}]$ and $[P, a^\dagger_\mathbf{p}]$. Discuss the particle content of the theory. Show that the Klein–Gordon field obeys Bose–Einstein statistics.

2

The Lagrangian density for a scalar particle interacting with a fermion is

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \phi \partial^\mu \phi \;-\; \frac{1}{2} m^2 \phi^2 \;+\; \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \;-\; g\, \bar{\psi}\phi\psi \,.$$

Write down the field equations satisfied by ψ, $\bar{\psi}$ and ϕ.

State the Feynman rules. Draw the tree-level Feynman diagrams for the processes $\psi\psi \to \psi\psi$, $\psi\bar{\psi} \to \psi\bar{\psi}$ and $\psi\phi \to \psi\phi$. Write down the respective scattering amplitudes.

3

The Lagrangian for a scalar field of charge e interacting with the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - m^2|\phi|^2$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + ieA_\mu$. Show that this Lagrangian has a gauge symmetry. Draw the two interaction vertices in this theory, identifying the corresponding interaction terms in the Lagrangian.

When one quantises the theory in Coulomb gauge, $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, the naive photon propagator is

$$D_{\mu\nu}(p) = \begin{cases} \dfrac{i}{p^2+i\epsilon}\left(\delta_{ij} - \dfrac{p_i p_j}{|\mathbf{p}|^2}\right) & \begin{matrix}\mu = i \neq 0\\ \nu = j \neq 0\end{matrix} \\[2ex] \dfrac{i}{|\mathbf{p}|^2} & \mu,\nu = 0 \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Draw the leading order diagrams for $\phi\bar{\phi} \to \phi\bar{\phi}$ scattering. Show that your answer can be expressed in terms of the Lorentz invariant propagator

$$D_{\mu\nu}(p) = -i\frac{g_{\mu\nu}}{p^2}$$

suitably contracted with external momenta.

4

Write an essay on symmetries in field theory. Your essay should include a statement and proof of Noether's theorem; give examples of important symmetries in different field theories; describe the difference between a global symmetry and a gauge symmetry.

END OF PAPER