UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 9:00 am to 11:00 am

PAPER 40

OPTIMAL INVESTMENT

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An agent may invest in a bank account paying interest at a constant continuously-compounded rate r, and in a risky asset, whose price S_t at time t evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

where σ and μ are constants, and W is a standard Brownian motion. If he chooses to invest wealth θ_t at time t in the risky asset, and to withdraw cash for consumption at rate c_t at time t, write down the evolution of the wealth w_t of the agent at time t. What conditions should θ and c satisfy?

Suppose now that his objective is to obtain

$$V(\xi, w) = \sup_{c \geqslant 0, \theta} E\left[\int_0^\infty e^{-\rho t} U(\xi_t)\, dt \,\middle|\, w_0 = w,\ \xi_0 = \xi \right].$$

where w is constrained to remain non-negative for all time, U is C^2 strictly increasing and strictly concave, and ξ is related to c by

$$\xi_t = e^{-\lambda t}\xi_0 + \int_0^t e^{-\lambda(t-s)} c_s\, ds.$$

Show that ξ satisfies the SDE

$$d\xi_t = (c_t - \lambda \xi_t)dt.$$

Explaining briefly, find the Hamilton-Jacobi-Bellman (HJB) equation for this problem. Assuming that U has the CRRA form $U'(x) = x^{-R}$ for some positive R not equal to 1, simplify the HJB equation, and explain why you expect that optimal behaviour will require that $c_t = 0$ whenever $w_t/\xi_t < x_*$ for some positive x_* (which you are not required to identify).

2

Suppose that W is a standard Brownian motion, and that X is the unique solution of the SDE

$$dX_t = \alpha(X_t)dW_t + \beta(X_t)dt,$$

where α, β are Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$. The price S_t at time t of a risky asset evolves according to

$$dS_t = S_t\{\, \sigma(X_t)dW_t + \mu(X_t)\, dt\ \}$$

where σ and μ are bounded functions from $\mathbb{R}$ to $\mathbb{R}$. An investor is able to invest in a riskless bank account paying interest at constant continuously-compounded rate r, and in the risky asset S. Suppose that his objective is to obtain

$$V(w,x) = \sup_{c\geqslant 0,\theta} E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \,\middle|\, w_0 = w,\ X_0 = x\right].$$

where $U'(x) = x^{-R}$ for some $R > 0$ different from 1, c is the rate of consumption chosen, and θ is his holding of the risky asset, supposing that wealth is required to remain non-negative at all times.

Briefly explaining your derivation, find an equation satisfied by V, and explain how it can be simplified because of the special form assumed for U. Express the equation for V after this simplification.

Suppose now that

$$\mu(x) = r + \sigma(x)\kappa$$

for some constant κ. Show that the equation for the value function V simplifies further, and explain the form you find.

3

An agent may invest in a bank account paying interest at a constant continuously-compounded rate r, and in a risky asset, whose price S_t at time t evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

where σ and μ are constants, and W is a standard Brownian motion. He has initially wealth x_0, but he borrows at time 0 a fixed amount D of money, thereby raising his available wealth for investment to $w_0 = x_0 + D$. He must pay interest at rate $\bar{r} > r$ on the money borrowed. In terms of his rate c_t of consumption withdrawal, and the wealth θ_t invested in the risky asset, write down the evolution of his available wealth w_t at time t.

Now suppose that $T \equiv \inf\{t : w_t \leqslant 0\}$ is the first time his available wealth falls to zero, and that his objective is to obtain

$$V(w) = \sup_{c \geqslant 0, \theta} E\left[\int_0^T e^{-\rho t} U(c_t)\, dt \,\middle|\, w_0 = w \right].$$

where $U : \mathbb{R}^+ \to \mathbb{R}^+$ is concave, strictly increasing, with $U(0) = 0$, $U'(0) = \infty$, $U'(\infty) = 0$. Write down the Hamilton-Jacobi-Bellman equation for V.

Assuming that $U(x) = x^{1-R}/(1-R)$ for some $R \in (0,1)$, find the dual form of the HJB equation and solve it as completely as you can, taking care to explain what are the appropriate boundary conditions.

[You may assume that $\gamma_M \equiv \{\rho + (R-1)(r + \frac{1}{2}\kappa^2/2R)\}/R > 0$, where $\kappa = (\mu - r)/\sigma$, and you may use the fact that $Q(1 - R^{-1}) = -\gamma_M$, where Q is the quadratic $Q(t) = \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$.]

4

An agent may invest in a bank account paying interest at a constant continuously-compounded rate r, and in a risky asset, whose price S_t at time t evolves as

$$dS_t = S_t(\sigma dW_t + \mu dt),$$

where σ and μ are constants, and W is a standard Brownian motion. What is the *state-price density process* ζ for this problem? Briefly explain its rôle in pricing contingent claims.

If w_t denotes his wealth at time t, the agent's objective is to obtain

$$\sup E[\,U(w_T)\,] \quad \text{subject to} \quad w_T - w_0 \geqslant \alpha(S_T - S_0),$$

where $\alpha \geqslant 0$ is fixed. Find the largest value $\bar{\alpha}$ of α for which the constraint can be satisfied, and explain how the agent would invest if $\alpha = \bar{\alpha}$.

If $\alpha < \bar{\alpha}$, show that the optimal terminal wealth w_T^* can be expressed as

$$w_T^* = \max\{\,\xi, I(\lambda\zeta_T)\,\}$$

for a random variable ξ which you should identify, and a positive scalar λ. Here, I is the inverse marginal utility: $U'(I(x)) = x$ for all $x > 0$. You should explain how the scalar λ is characterized.

END OF PAPER