UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2013 9:00 am to 12:00 pm

PAPER 39

ADVANCED FINANCIAL MODELS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let

$$dX_t = -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t^X$$
$$d\sigma_t = A(\sigma_t)dt + B(\sigma_t)dW_t^\sigma$$

where W^X and W^σ are Brownian motions with correlation ρ. Fix a time horizon $T > 0$ and a function g, and let the smooth function U on $[0,T] \times \mathbb{R} \times \mathbb{R}$ solve the PDE

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial \sigma} + \frac{1}{2}B^2\frac{\partial^2 U}{\partial \sigma^2} + \sigma B\rho\frac{\partial^2 U}{\partial \sigma \partial X} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial X}\right) = 0$$

with boundary condition

$$U(T, \sigma, X) = g(X) \text{ for all } \sigma, X \in \mathbb{R}\,.$$

(a) Show that the process $(M_t)_{0 \leqslant t \leqslant T}$ defined by $M_t = U(t, \sigma_t, X_t)$ is a local martingale.

Now suppose that $g(X) = e^{\theta X}$ for a constant θ.

(b) By making the substitution $U(t, \sigma, X) = e^{\theta X}V(t, \sigma)$ derive a PDE for V. What is the boundary condition?

Specialise to the case where $A(\sigma) = (a - \sigma)\lambda$ and $B(\sigma) = b$ for some constants a, λ and b.

(c) Show there is a solution to the PDE derived in part (b) of the form

$$V(t, \sigma) = e^{P(T-t)+Q(T-t)\sigma+R(T-t)\sigma^2}$$

for function P, Q and R satisfying a system of ordinary differential equations

$$\dot{R} = F(R)$$
$$\dot{Q} = G(Q, R)$$
$$\dot{P} = H(Q, R)$$

where $\dot{R}$ denotes the derivative of R, etc., and the functions F, G and H should be given explicitly in terms of the parameters a, λ, b, ρ and θ.

[You may use standard facts from stochastic calculus without proof, as long as they are clearly stated.]

2

Let S be a non-negative random variable with $\mathbb{E}(S) < \infty$, and let

$$C(K) = \mathbb{E}[(S-K)^+] \text{ and } M(p) = \mathbb{E}[S^p]$$

for all $K \geqslant 0$ and $p \geqslant 1$.

(a) Show the identity

$$S^{1+\varepsilon} = (1+\varepsilon)\varepsilon \int_0^\infty K^{\varepsilon-1}(S-K)^+ dK$$

for all $\varepsilon > 0$.

(b) Show that if there exists $\delta > 0$ such that $\sup_{K \geqslant 0} K^\delta C(K) < \infty$ then $M(1+\varepsilon) < \infty$ for all $0 \leqslant \varepsilon < \delta$.

(c) Show that

$$S^{1+\varepsilon} \geqslant \frac{(1+\varepsilon)^{1+\varepsilon}}{\varepsilon^\varepsilon} K^\varepsilon (S-K)^+$$

for all $K \geqslant 0$ and $\varepsilon > 0$.

[*Hint: Consider the minimum of the function* $f(x) = x^{1+\varepsilon} - ax$, *where* $a = \frac{(1+\varepsilon)^{1+\varepsilon}}{\varepsilon^\varepsilon} K^\varepsilon$.]

(d) Show that if $M(1+\varepsilon) < \infty$ then $\sup_{K \geqslant 0} K^\varepsilon C(K) < \infty$.

3

Consider a one-period model $(P_t)_{t \in \{0,1\}}$ of a market with n assets.

(a) What does it mean to say a contingent claim can be replicated? What does it mean to say that the market is complete?

Now suppose that the market is complete.

(b) Show that the sample space can be partitioned into at most n events of positive probability.

(c) Suppose that $P_0 = \mathbb{E}(XP_1) = \mathbb{E}(YP_1)$ for positive random variables X and Y. Show that $X = Y$ almost surely.

(d) Let Q be the $n \times n$ matrix $Q = \mathbb{E}(P_1 P_1^{\mathrm{T}})$, where the notation A^{T} denotes the transpose of the matrix A. Suppose that Q is positive definite, and let $Z = P_0^{\mathrm{T}} Q^{-1} P_1$. Show that $\xi_0 = \mathbb{E}(Z\xi_1)$ is the cost of replicating the contingent claim with payout ξ_1. Find a random vector W such that the replicating portfolio can be expressed as $H = \mathbb{E}[W\xi_1]$.

4

Let $(Y_t)_{0 \leqslant t \leqslant T}$ be an integrable discrete-time process, adapted to a filtration $(\mathcal{F}_t)_{0 \leqslant t \leqslant T}$ where $\mathcal{F}_0$ is trivial. Define the Snell envelope process by $U_T = Y_T$ and

$$U_t = \max\{Y_t, \mathbb{E}(U_{t+1}|\mathcal{F}_t)\} \text{ for } 0 \leqslant t < T\,.$$

(a) Show that $U_0 \geqslant \mathbb{E}(Y_\tau)$ for any stopping time τ.

(b) Let $\tau_* = \inf\{t \geqslant 0 : U_t = Y_t\}$. Show that $U_0 = \mathbb{E}(Y_{\tau_*})$.

(c) Consider the following game. In round n, the player is offered ξ_n units of money. He can either accept the offer, at which point the game ends, or he can reject the offer and proceed to round $n+1$. The game lasts at most four rounds. Assume that the numbers $\xi_1, \xi_2, \xi_3, \xi_4$ are independent random variables, uniformly distributed on $[0,1]$, and the player's goal is to maximise his expected payout. Determine the player's optimal strategy explicitly.

5

Consider a discrete model of a market of N zero coupon bonds of maturities $1, 2, \ldots, N$ and unit principal value. Let $P_t(T)$ be the time-t price of the bond with maturity T. Assume $P_t(T) > 0$ for all $0 \leqslant t \leqslant T \leqslant N$, and that there is no arbitrage in this market.

(a) Explain why there exists a positive adapted process $(Z_t)_{0 \leqslant t \leqslant N}$ such that

$$P_t(T) = \frac{\mathbb{E}(Z_T | \mathcal{F}_t)}{Z_t} .$$

[You may use any result from the course without proof, as long as it is carefully stated.]

(b) Suppose that $T \mapsto P_t(T)$ is decreasing for all t. Prove that the process $(Z_t)_{0 \leqslant t \leqslant N}$ from (a) is a supermartingale.

(c) Let

$$r_t = \frac{1}{P_{t-1}(t)} - 1 .$$

Fix $T \leqslant N$ and consider a European contingent claim with time T payout $\xi_T = r_T - f$. Show that if

$$f = \frac{P_0(T-1)}{P_0(T)} - 1$$

then the initial no-arbitrage price is $\xi_0(T) = 0$. Find a replicating strategy for the claim.

(d) An interest rate swap is a contract that promises T payments of the amount

$$r_t - s$$

at each time $t = 1, \ldots, T$, where s is a constant. If the initial no-arbitrage price of this claim is zero, find s in terms of the initial bond prices $P_0(1), \ldots, P_0(T)$.

6

Consider a two asset model with price dynamics

$$\begin{aligned} dB_t &= B_t r_t dt \\ dS_t &= S_t(\mu_t dt + \sigma_t dW_t) \end{aligned}$$

where r, μ, σ are continuous processes such that $\sigma_t(\omega) > 0$ for all (t, ω) and where W is a Brownian motion. Suppose all processes are adapted to the filtration generated by W.

(a) Show that there is a unique state price density process $Z = (Z_t)_{t \geqslant 0}$ with $Z_0 = 1$, and that its dynamics are of the form

$$dZ_t = -Z_t(r_t dt + \lambda_t dW_t)$$

where the process λ is to be expressed in terms of the processes r, μ and σ.

(b) Let $V = (V_t)_{t \geqslant 0}$ be the wealth of a self-financing investor. Show that the process ZV is a local martingale. Show that if her initial wealth is $V_0 = 0$ and she invests so that her wealth $V_t \geqslant 0$ is non-negative at future times $t \geqslant 0$, then $V_t = 0$ almost surely for all $t \geqslant 0$.

Now consider a market where

$$\begin{aligned} B_t &= e^{2\sqrt{t}} \\ S_t &= e^{W_t - t/2}, \end{aligned}$$

where W is still denotes a Brownian motion and all processes are assumed adapted to the filtration generated by W.

(c) Show that there is no state price density process for this market model.

[You may use standard results from stochastic calculus as long as they are stated clearly.]

END OF PAPER