UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 1:30 pm to 4:30 pm

PAPER 38

MATHEMATICS OF OPERATIONAL RESEARCH

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & -\sum_{i=1}^{n} \log(\alpha_i + x_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 1 \\ & x \geqslant 0, \end{aligned}$$

where $\alpha_i > 0$ for all $i = 1, \ldots, n$.

(a) Solve this problem using the method of Lagrange multipliers. Show that the optimal solution is unique and explain how it can be found.

(b) Find the optimal solution for the case where $n = 4$ and

$$\alpha = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{5}, \frac{3}{4}\right).$$

2

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 4x_2 + 3x_3 + x_4 \\ \text{subject to} \quad & x_1 + 2x_2 - x_3 \geqslant 3 \\ & 2x_2 + x_3 - 3x_4 \geqslant 4 \\ & -2x_1 + 3x_3 + x_4 \geqslant 1, \\ & x_1, x_2, x_3, x_4 \geqslant 0. \end{aligned}$$

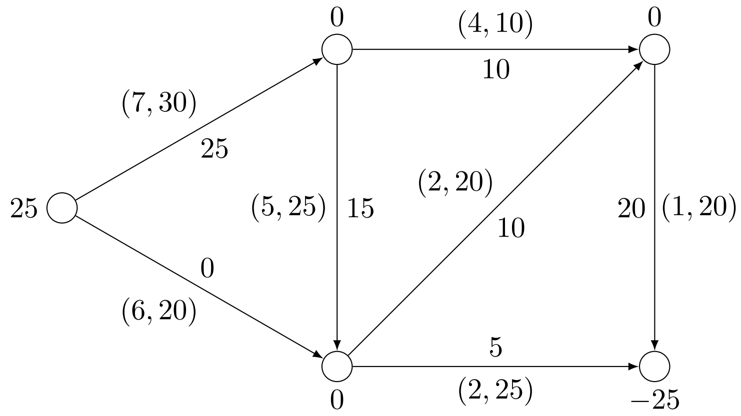
(a) Determine the dual problem with variables λ_1, λ_2, and λ_3.

(b) Solve the dual using the simplex method by starting from the feasible solution where $\lambda = (0, 0, 0)$ and pivoting in such a way that degenerate solutions are avoided whenever possible. Verify that $\lambda = (0, 2, 1/3)$ is an optimal solution and use this information to derive an optimal solution of the primal. Explain carefully what you are doing.

(c) Prove from first principles that the solution to the primal is indeed optimal.

3

Consider the following minimum-cost flow problem with lower bounds equal to zero, where vertex i is labeled with b_i and edge (i,j) with the pair $(c_{ij}, \overline{m}_{ij})$ and a flow x_{ij}:



(a) Use the network simplex method to find an optimal solution to this problem. Explain carefully what you are doing. Why can the flow vector x be used as an initial solution?

(b) Derive the appropriate optimality conditions and use them to show that the solution you found is indeed optimal.

4

A vertex cover of an undirected graph (V, E) is a set of vertices that contains at least one endpoint of every edge, i.e., a set $U \subseteq V$ such that for every $\{u, v\} \in E$, $u \in U$ or $v \in U$.

(a) Show that it is NP-hard to decide whether a graph has a vertex cover of a given size k. To this end, consider an instance of the NP-complete satisfiability problem for Boolean formulae in conjunctive normal form with three literals per clause. Represent each variable x by two vertices x and $\bar{x}$ that are connected by an edge, and each clause by three vertices that are all connected by edges. Furthermore, if variable x occurs as a positive or negative literal in a given clause, then respectively connect vertex x or $\bar{x}$ to the corresponding vertex for the clause. Now show that the graph has a vertex cover of size $n + 2m$ if and only if the Boolean formula is satisfiable, where n is the number of variables in the Boolean formula and m is the number of clauses.

(b) Use the max-flow min-cut theorem to show that in any bipartite graph, the size of a maximum-cardinality matching equals the size of a minimum-cardinality vertex cover. Argue that in bipartite graphs, a minimum vertex cover can be found in polynomial time in the number of vertices and edges. State clearly any results that you use.

5

A bimatrix game with payoff matrices P and Q is called symmetric if $P = Q^T$. A strategy profile (s, t) is called symmetric if $s = t$, and an equilibrium is called symmetric if it is a symmetric strategy profile.

(a) Show that every symmetric bimatrix game has a symmetric equilibrium. To this end you may modify the proof of Nash's theorem, but if you do you must also reproduce the parts of the proof that remain unchanged.

Consider the non-degenerate bimatrix game with payoff matrices

$$P = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 2 & 2 \\ 3 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 2 & 0 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

(b) Find all three equilibria of this game.

(c) Argue that every non-degenerate symmetric game must have an odd number of symmetric equilibria. Verify that this is true for the game above.

6

(a) Define the Nash bargaining solution and show that it is characterized by Pareto efficiency, symmetry, invariance under positive affine transformations, and independence of irrelevant alternatives.

(b) Determine the Nash bargaining solution for the bargaining problem (F, d), where F is the convex hull of payoff vectors, and d is the vector of security level payoffs, of the bimatrix game with payoff matrices

$$P = \begin{pmatrix} 0 & 4 \\ 3 & 2 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 0 & 0 \\ 2 & 3 \end{pmatrix}.$$

Explain carefully what you are doing.

(c) Determine the Nash bargaining solution for the bargaining problem defined as above for payoff matrices P and Q^T.

END OF PAPER