MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 1:30 pm to 4:30 pm

PAPER 37

STOCHASTIC NETWORKS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write an essay on mathematical models of loss networks. Your essay should cover the following topics, but need not be restricted to them.

(i) The stationary distribution for a loss network operating under fixed routing.

(ii) The Erlang fixed point approximation for a loss network, and its uniqueness when routing is fixed.

(iii) An example of a loss network with alternative routing where the Erlang fixed point approximation is not unique.

2

Write an essay on mathematical models of traffic flow through networks. Your essay should cover the following topics, but need not be restricted to them.

(a) The definition of a Wardrop equilibrium.

(b) The relationship between a Wardrop equilibrium and optimization formulations of network flow.

(c) Braess' paradox.

3

Outline a mathematical model of a slotted infinite-population random access scheme, where N_t is the number of stations with a packet to transmit and each such station independently transmits its packets with probability $1/S_t$. Interpret the equation

$$N_{t+1} = N_t + Y_t - I[Z_t = 1]$$

where $Z_t = 0, 1$ or $*$ according as $0, 1$ or more than 1 packets are transmitted in slot $(t, t+1)$ and Y_t is the number of arrivals in slot $(t, t+1)$, assumed to have a Poisson distribution with mean ν.

Suppose that S_t is updated by the recursion

$$S_{t+1} = \max\{1, S_t + aI[Z_t = 0] + bI[Z_t = 1] + cI[Z_t = *]\}$$

for a triplet (a, b, c). Is (S_t, N_t) a Markov chain?

Motivate the differential equation

$$\frac{ds}{dt} = (a-c)e^{-\kappa} + (b-c)\kappa e^{-\kappa} + c \qquad \frac{dn}{dt} = \nu - \kappa e^{-\kappa}$$

where $\kappa = n/s$ in terms of the expected drift of (N_t, S_t).

Find conditions on (a, b, c) such that, provided $\nu < e^{-1}$, any trajectory solving the differential equations converges to the origin. What happens if $\nu > e^{-1}$?

4

Derive Chernoff's bound

$$P\{Y \geqslant 0\} \leqslant \inf_{s \geqslant 0} \mathbb{E}[e^{sY}]\,.$$

Let

$$X = \sum_{j=1}^{J} \sum_{i=1}^{n_j} X_{ji}$$

where X_{ji} are independent random variables with

$$\alpha_j(s) = \frac{1}{s} \log \mathbb{E}[e^{sX_{ji}}]\,,$$

for $i = 1, 2, \ldots, n_j$. Show that

$$\sum_{j=1}^{J} n_j \alpha_j(s) \leqslant C - \frac{\gamma}{s} \quad \Rightarrow \quad P\{X \geqslant C\} \leqslant e^{-\gamma}\,,$$

and briefly discuss the interpretation of $\alpha_j(s)$ as an *effective bandwidth*.

In the case where $X_{ji} \sim N(\lambda_j, \sigma_j^2)$, show that the above implication can be written in the form

$$\sum_{j=1}^{J} n_j \lambda_j + \left(2\gamma \sum_{j=1}^{J} n_j \sigma_j^2 \right)^{1/2} \leqslant C \quad \Rightarrow \quad P\{X \geqslant C\} \leqslant e^{-\gamma}\,.$$

Under the same distributional assumptions on X_{ji}, determine a necessary and sufficient condition on $n_1, n_2, \ldots, n_J$ such that $P\{X \geqslant C\} \leqslant e^{-\gamma}$, and show that it can be written in the form

$$\sum_{j=1}^{J} n_j \lambda_j + \phi \left(\sum_{j=1}^{J} n_j \sigma_j^2 \right)^{1/2} \leqslant C$$

for a constant ϕ to be determined.

5

Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation, $(x_r, r \in R)$.

Consider a *linear* network with resources $J = \{1, 2, \ldots, I\}$, each of unit capacity, and routes $R = \{0, 1, 2, \ldots, I\}$ where we use the symbol 0 to represent a route $\{1, 2, \ldots, I\}$ which traverses the entire set of resources, and the symbol i to represent a route $\{i\}$ through a single resource, for $i = 1, 2, \ldots, I$. Show that under a proportionally fair rate allocation

$$x_0 n_0 + x_i n_i = 1 \quad \text{if} \quad n_i > 0, \quad i = 1, 2, \ldots, I$$

and

$$x_0 = \frac{1}{n_0 + \sum_{i=1}^{I} n_i} \quad \text{if} \quad n_0 > 0\,.$$

Suppose now that flows describe the transfer of documents through the linear network above, that new flows originate as independent Poisson processes of rates $\nu_r, r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ take the form

$$\pi(n) = (1 - \rho_0)^{1-I} \prod_{i=1}^{I} (1 - \rho_0 - \rho_i) \begin{pmatrix} \sum_{r=0}^{I} n_r \\ n_0 \end{pmatrix} \prod_{r=0}^{I} \rho_r^{n_r}\,,$$

provided $\rho_0 + \rho_i < 1$, $i = 1, 2, \ldots, I$ where $\rho_r = \nu_r / \mu_r$.

Find $\mathbb{E}(n_i)$, under the distribution π, for $i = 1, 2, \ldots, I$.

END OF PAPER