UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2013 1:30 pm to 4:30 pm

PAPER 36

CONTEMPORARY SAMPLING TECHNIQUES AND COMPRESSED SENSING

You may attempt ***ALL*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A matrix $A \in \mathbb{C}^{m \times N}$ is said to satisfy the null space property relative to a set $S \subset \{1, \ldots, N\}$ if

$$\|v_S\|_1 < \|v_{S^c}\|_1, \qquad \text{for all}\, v \in \mathcal{N}(A) \setminus \{0\}.$$

It is said to satisfy the null space property of order s if it satisfies the null space property relative to any set $S \subset \{1, \ldots, N\}$ with $|S| \leqslant s$. ($\mathcal{N}(A)$ denotes the null space of A, $S^c = \{1, \ldots, N\} \setminus S$ and v_S denotes the vector equal to v on S and zero on S^c).

(a) Given a matrix $A \in \mathbb{C}^{m \times N}$, show that every s-sparse vector $x \in \mathbb{C}^N$ (meaning that x has at most s non-zero coefficients) is the unique solution to

$$\min \|z\|_1 \quad \text{subject to} \quad Az = Ax$$

if and only if A satisfies the null space property of order s.

(b) Show that every s-sparse vector $x \in \mathbb{C}^N$ is the unique solution to

$$\min \|z\|_0 \quad \text{subject to} \quad Az = Ax$$

if A satisfies the null space property of order s (here the l^0 "norm" denotes the number of non-zero elements in the vector).

In the next part you may use the following fact without proof: Given a matrix $A \in \mathbb{C}^{m \times N}$ and $0 < r < 1$, every s-sparse vector $x \in \mathbb{C}^N$ is the unique solution to

$$\min \|z\|_r \quad \text{subject to} \quad Az = Ax \tag{1}$$

if and only if, for any set $S \subset \{1, \ldots, N\}$ with $|S| \leqslant s$,

$$\|v_S\|_r < \|v_{S^c}\|_r, \quad \forall v \in \mathcal{N}(A) \setminus \{0\}.$$

(c) Let $0 < p < q < 1$. Show that if every s-sparse vector $x \in \mathbb{C}^N$ is the unique solution to (1) with $r = q$ then every s-sparse vector $x \in \mathbb{C}^N$ is also the unique solution to (1) with $r = p$.

2

The s-th restricted isometry constant $\delta_s = \delta_s(A)$ of a matrix $A \in \mathbb{C}^{m\times N}$ is the smallest $\delta \geqslant 0$ such that

$$(1-\delta)\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\delta)\|x\|_2^2$$

for all s-sparse vectors $x \in \mathbb{C}^n$.

(a) Let $A \in \mathbb{C}^{m\times N}$. Show that

$$\delta_s = \max_{S\subset\{1,\dots,N\},|S|\leqslant s} \|A_S^* A_S - Id\|_{2\to 2},$$

where A_S denotes the submatrix of A obtained by selecting the columns indexed by S.

Let $A \in \mathbb{C}^{m\times N}$ be a matrix with l^2-normalised columns $a_1,\dots,a_N$, i.e. $\|a_i\|_2 = 1$ for all $1 \leqslant i \leqslant N$. The coherence $\mu = \mu(A)$ of the matrix A is defined as

$$\mu = \max_{1\leqslant i\neq j\leqslant N} |\langle a_i, a_j\rangle|.$$

Also, the l^1-coherence function μ_1 of the matrix A is defined, for $1 \leqslant s \leqslant N-1$ by

$$\mu_1(s) = \max_{i\in\{1,\dots,N\}} \max\left\{\sum_{j\in S} |\langle a_i, a_j\rangle|, S\subset\{1,\dots,N\}, |S| = s, i\notin S\right\}.$$

In the following problems $A \in \mathbb{C}^{m\times N}$ with l^2-normalised columns.

(b) Let $1 \leqslant s \leqslant N$. Show that for all s-sparse vectors $x \in \mathbb{C}^N$

$$(1-\mu_1(s-1))\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\mu_1(s-1))\|x\|_2^2.$$

(You may use the fact that if $B \in \mathbb{C}^{n\times n}$ and λ is an eigenvalue, then there exists an index $j \in \{1,\dots,n\}$ such that

$$|\lambda - B_{jj}| \leqslant \sum_{l\in\{1,\dots,n\}\setminus\{j\}} |B_{jl}|,$$

without proving this result.)

(c) Show that

$$\delta_1 = 0, \quad \delta_2 = \mu, \qquad \delta_s \leqslant \mu_1(s-1) \leqslant (s-1)\mu, \quad s \geqslant 2.$$

3

Let $A \in \mathbb{R}^{m \times N}$ with columns $a_1, \ldots, a_N$ and $x \in \mathbb{R}^N$ such that $\mathrm{supp}(x) = S \subset \{1, \ldots, N\}$. Consider the three conditions:

(i) $|\sum_{j \in S} \overline{\mathrm{sgn}(x_j)} v_j| < \|v_{S^c}\|_1$ for all $v \in \mathcal{N}(A) \setminus \{0\}$,

(ii) A_S is injective and there exists a vector $h \in \mathbb{R}^m$ such that

$$(A^* h)_j = \mathrm{sgn}(x_j), \quad j \in S, \qquad |(A^* h)_l| < 1, \quad l \in S^c,$$

(iii) A_S is injective and

$$|\langle (A_S^* A_S)^{-1} A_S^* a_l, \mathrm{sgn}(x_S) \rangle| < 1, \quad \text{for all}\, l \in S^c,$$

(in this case x_S is understood to be in $\mathbb{R}^s$, where $s = |S|$).

This problem is about relating the above conditions to the optimisation problem

$$\min \|z\|_{l^1} \quad \text{subject to} \quad Az = Ax. \tag{1}$$

(a) Show that x is the unique minimiser of (1) when condition (i) holds.

(b) Show that if x is the unique minimiser of (1) then (i) holds.

(c) Show that x is the unique minimiser of (1) when condition (ii) holds.

(d) Show that x is the unique minimiser of (1) when condition (iii) holds.

4

Let $\mathcal{V}, \mathcal{W}$ be closed subspaces of the Hilbert space $\mathcal{H}$, and let $P_{\mathcal{V}}$ denote the projection onto $\mathcal{V}$. The angle $\theta_{\mathcal{V},\mathcal{W}} \in [0, \pi/2]$ is defined by

$$\cos\left(\theta_{\mathcal{V},\mathcal{W}}\right) := \inf_{v \in \mathcal{V}, \|v\|=1} \|P_{\mathcal{W}} v\|.$$

In the problems below you may use the following without a proof: If $\mathcal{W} \oplus \mathcal{V} = \mathcal{H}$, then

$$\|P_{\mathcal{V}\mathcal{W}}\| = \|I - P_{\mathcal{V}\mathcal{W}}\| = \sec(\theta_{\mathcal{V},\mathcal{W}^{\perp}}),$$

where $P_{\mathcal{V}\mathcal{W}}$ denotes the oblique projection of $\mathcal{H}$ on $\mathcal{V}$ along $\mathcal{W}$.

(a) Let $\mathcal{V}, \mathcal{W}$ be closed subspaces of $\mathcal{H}$. Show that if $\cos\left(\theta_{\mathcal{W},\mathcal{V}^{\perp}}\right)$ and $\cos\left(\theta_{\mathcal{V}^{\perp},\mathcal{W}}\right)$ are both positive then $\mathcal{W} \oplus \mathcal{V} = \mathcal{H}$.

(b) Let $\mathcal{U}$ and $\mathcal{V}$ be closed subspaces of $\mathcal{H}$ such that $\cos\left(\theta_{\mathcal{U},\mathcal{V}^{\perp}}\right) > 0$. Suppose also that $\dim(\mathcal{U}) = \dim(\mathcal{V}^{\perp}) = n < \infty$. Show that $\mathcal{U} \oplus \mathcal{V} = \mathcal{H}$.

(c) Let $\{\phi_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{\in\mathbb{N}}$ be orthonormal systems of $\mathcal{H}$ and let $\mathcal{T}_n = \mathrm{span}\{\phi_1, \ldots, \phi_n\}$ and $\mathcal{S}_n = \mathrm{span}\{\psi_1, \ldots, \psi_n\}$. Suppose that

$$\cos(\theta_{\mathcal{T}_n,\mathcal{S}_n}) > 0.$$

Show that for any $f \in \mathcal{H}$, there exists a unique $\tilde{f}_n \in \mathcal{T}_n$ satisfying

$$\langle \tilde{f}_n, \psi_j \rangle = \langle f, \psi_j \rangle, \quad j \in \{1, \ldots, n\}.$$

Furthermore, show that the following bounds hold:

$$\|\tilde{f}_n\| \leqslant \sec(\theta_{\mathcal{T}_n,\mathcal{S}_n})\|f\|, \quad \|f - P_{\mathcal{T}_n} f\| \leqslant \|f - \tilde{f}_n\| \leqslant \sec(\theta_{\mathcal{T}_n,\mathcal{S}_n})\|f - P_{\mathcal{T}_n} f\|.$$

END OF PAPER