UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 10 June, 2013 9:00 am to 12:00 pm

PAPER 3

TOPICS IN GROUP THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a finite group.

(a) Define what it means for G to be *soluble*.

(b) Show that if G is soluble then so is any subgroup or quotient of G.

(c) Show that if G is soluble then a minimal normal subgroup of G is an elementary abelian p-group for some prime p.

(d) Let π be a set of primes. Define what it means for a subgroup of G to be a *Hall π-subgroup* of G.

(e) Prove that if G is soluble and π is a set of primes, then G has a Hall π-subgroup. [You may assume without proof the Frattini argument, which states that if $N \trianglelefteq G$ and P is a Sylow p-subgroup of N for some prime p, then $G = N_G(P)N$.]

(f) Let $|G| = pqr$, where p, q and r are primes with $p > q > r$. Show that G has a normal Sylow p-subgroup and a normal Hall $\{p, q\}$-subgroup, and is soluble.

2 Let Ω be a finite set, and (G, Ω) be a transitive permutation group. Set $\Omega^+ = \Omega \cup \{\omega\}$ for some $\omega \notin \Omega$.

(a) Define what it means for a permutation group (G^+, Ω^+) to be a *one-point extension* of (G, Ω).

(b) Suppose $G^+ = \langle G, x\rangle$ where $x \in S_{\Omega^+}$ interchanges ω and some $\alpha \in \Omega$. State a result giving necessary and sufficient conditions for (G^+, Ω^+) to be a one-point extension of (G, Ω).

(c) For $t \in \mathbb{N}$, define what it means for G to be *sharply t-transitive* on Ω.

(d) Show that if G is sharply 2-transitive on Ω, then G has a regular characteristic subgroup.

(e) State Iwasawa's Lemma.

(f) Let $G = \langle a, b\rangle < S_8$, where $a = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$, $b = (2\ 3\ 5\ 4\ 7\ 8\ 6)$.

(i) Show that G is sharply 2-transitive on $\Omega = \{1, \ldots, 8\}$, and identify its regular characteristic subgroup.

(ii) By finding an element $x \in S_9$ satisfying the conditions of the result in (b), show that (G, Ω) has a one-point extension (G^+, Ω^+).

(iii) Prove that G^+ is a simple group.

3 Let $V = V_{2m}(q)$ be a vector space of dimension $2m$ over a field $\mathbb{F}_q$, and $(\ ,\ )$ be a non-degenerate alternating bilinear form on V. Let $e_1, f_1, e_2, f_2, \ldots, e_m, f_m$ be a symplectic basis of V; write matrices of $\mathrm{GL}(V)$ with respect to the ordered basis $e_1, e_2, \ldots, e_m, f_m, \ldots, f_2, f_1$.

(a) Define the *symplectic group* $\mathrm{Sp}(V)$, and determine its order.

(b) Define the following elements of $\mathrm{GL}(V)$: given $\lambda \in \mathbb{F}_q$,

$$\begin{array}{lll} \text{for } 1 \leqslant i < j \leqslant m, & x_{ij}(\lambda) \text{ sends } & e_i \mapsto e_i + \lambda e_j,\ f_j \mapsto f_j - \lambda f_i, \\ \text{for } 1 \leqslant i < j \leqslant m, & y_{ij}(\lambda) \text{ sends } & e_i \mapsto e_i + \lambda f_j,\ e_j \mapsto e_j + \lambda f_i, \\ \text{for } 1 \leqslant i \leqslant m, & z_i(\lambda) \text{ sends } & e_i \mapsto e_i + \lambda f_i, \end{array}$$

with all other basis vectors being fixed in each case. Prove that each of these elements lies in $\mathrm{Sp}(V)$, and that the group which they generate is a Sylow p-subgroup of $\mathrm{Sp}(V)$, where p is the prime dividing q. [Hint: it may help to start by considering the group generated by the elements with $i = 1$.]

(c) Take $1 \leqslant k < m$ and set $W = \langle f_1, \ldots, f_k \rangle$, $W' = \langle e_1, \ldots, e_k \rangle$ and $U = \langle e_{k+1}, f_{k+1}, \ldots, e_m, f_m \rangle$, so that $V = W \oplus W' \oplus U$ and $W^\perp = W \oplus U$. Consider the flag $0 \subset W \subset W^\perp \subset V$; let its stabilizer in $\mathrm{Sp}(V)$ be P. Thus each element of P is a matrix of the form

$$\begin{pmatrix} A & D & F \\ 0 & B & E \\ 0 & 0 & C \end{pmatrix}$$

where $A \in \mathrm{GL}(W')$ and $C \in \mathrm{GL}(W)$ are $k \times k$ and $B \in \mathrm{GL}(U)$ is $(2m-2k) \times (2m-2k)$; for convenience write the above matrix as (A, B, C, D, E, F). Let Q be the subgroup consisting of matrices in P of the form (I, I, I, D, E, F) (where I is the identity matrix of the relevant size) and L be the subgroup consisting of matrices in P of the form $(A, B, C, 0, 0, 0)$.

(i) Show that $|Q| = q^{2k(m-k)+k(k+1)/2}$.

(ii) Show that for all $C \in \mathrm{GL}(W)$ there is a unique $A \in \mathrm{GL}(W')$ such that $(A, I, C, 0, 0, 0) \in L$; deduce that $L \cong \mathrm{GL}_k(q) \times \mathrm{Sp}_{2m-2k}(q)$.

(iii) Show that $P = Q : L$, and hence compute $|P|$; check that your answer agrees with that obtained by counting the number of ways of choosing the vectors $f_1, \ldots, f_k$ in a symplectic basis of V and using the Orbit-Stabilizer Theorem.

4 Let H and H' be two disjoint 6-sets (i.e., sets of size 6).

(a) Define what are meant by *points*, *duads*, *synthemes* and *totals* of H. Prove that H has 6 points, 15 duads, 15 synthemes and 6 totals.

(b) Prove that a bijection from points of H to totals of H' may naturally be extended to a bijection from duads of H to synthemes of H', from synthemes of H to duads of H', from totals of H to points of H', and from $3|3$ partitions of H to $3|3$ partitions of H'.

(c) Define what is meant by a *Steiner system* $S(l, m, n)$, where $l, m, n \in \mathbb{N}$ with $2 \leqslant l < m < n$. Given a map Θ as in (b), exhibit a Steiner system $S(5, 6, 12)$ on $H \cup H'$, proving that the sets you take have the properties required.

END OF PAPER