UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 1:30 pm to 3:30 pm

PAPER 29

TIME SERIES AND MONTE CARLO INFERENCE

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Time Series

A stationary process $\boldsymbol{X} = (X_t)$ is related to $\boldsymbol{\epsilon} = (\epsilon_t)$, a white-noise process with variance σ^2, by

$$X_t = \sum_{r=-\infty}^{\infty} a_r \epsilon_{t-r} \qquad (*)$$

and this relationship can be inverted, as

$$\epsilon_t = \sum_{r=-\infty}^{\infty} b_r X_{t-r}. \qquad (**)$$

Under what conditions on the coefficients in $(*)$ and $(**)$ is this relationship:

(i) causal?

(ii) invertible?

Consider an ARMA(p, q) process given by

$$\phi(B) X_t = \theta(B) \epsilon_t \qquad (\dagger)$$

where $\phi(z) \equiv 1 - \sum_{r=1}^{p} \phi_r z^r$ and $\theta(z) \equiv 1 + \sum_{s=1}^{q} \theta_s z^s$ are polynomials, and B is the back-shift operator. Under what conditions is $(\dagger)$ a causal and invertible representation?

The stationary ARMA$(1,1)$ process $\boldsymbol{X}$ can be represented as

$$X_t = 2X_{t-1} + 2\epsilon_{t-1} + \epsilon_t. \qquad (\ddagger)$$

Is this representation causal and/or invertible?

Show that $\boldsymbol{X}$ has an alternative ARMA$(1,1)$ representation as

$$X_t = \tfrac{1}{2} X_{t-1} + \tfrac{1}{2} \eta_{t-1} + \eta_t$$

for some white-noise process $\boldsymbol{\eta}$ of variance σ^2.

Compute the best linear predictor of X_1, based on the semi-infinite history $(X_t : t \leqslant 0)$. [You are not required to prove convergence of infinite series.]

2 Time Series

Let $\boldsymbol{X} = (X_t)$ be a weakly stationary process with autocovariance function $\boldsymbol{\gamma} = (\gamma_r : r = 0, 1, \ldots)$. When does the *spectral density function* $f : [0, \pi] \to \mathbb{R}$ of $\boldsymbol{X}$ exist? For the case that it does, express $f(\omega)$ in terms of $\boldsymbol{\gamma}$, and γ_k in terms of f.

Show that, if $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$ (*i.e.*, $Z_t = X_t + Y_t$, all t), where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent stationary processes with respective spectral densities f_X and f_Y, then $\boldsymbol{Z}$ has spectral density $f_Z = f_X + f_Y$.

Suppose now that $\boldsymbol{X}$ has the ARMA$(1, 1; \theta, \phi)$ representation:

$$X_t = \phi X_{t-1} + \theta\epsilon_{t-1} + \epsilon_t$$

where the driving white noise process $\boldsymbol{\epsilon}$ has variance v; and that $\boldsymbol{Y}$ is white noise with variance w, independently of $\boldsymbol{X}$. What is the spectral density of $\boldsymbol{X}$? Find the spectral density of $\boldsymbol{Z} = \boldsymbol{X} + \boldsymbol{Y}$, and hence show that $\boldsymbol{Z}$ can be represented as ARMA$(1, 1; \alpha, \phi)$, where

$$\left(\frac{1+\alpha}{1-\alpha}\right)^2 = \frac{v(1+\theta)^2 + w(1-\phi)^2}{v(1-\theta)^2 + w(1+\phi)^2}.$$

Find the variance λ of the white noise process driving this representation, in terms of $(\theta, \phi, v, w, \alpha)$.

3 Monte Carlo Inference

Describe the *Box-Muller algorithm* for generating independent random variables from the standard normal distribution Norm$(0, 1)$, and explain why it works.

Explain how this algorithm could be applied to generate independent binary variables $Y_1, \ldots, Y_n$, such that $\Pr(Y_i = 1) = \Phi(x_i)$, where Φ is the standard normal distribution function and $x_1, \ldots, x_n$ are given constants.

Now suppose $\Pr(Y_i = 1) = \Phi(\beta x_i)$ where β is an unknown parameter which is assigned a Norm$(0, 1)$ prior distribution. We observe $Y_i = y_i$ $(i = 1, \ldots, n)$, and wish to simulate from the posterior distribution of β. Explain in detail how you could use data augmentation and Gibbs sampling to do this. [You need not compute the required distributions explicitly, but should indicate how you would simulate from them.]

4 Monte Carlo Inference

An observable X arises from one of a finite collection $\{M_j : j \in \mathcal{J}\}$ of possible models. The prior probability that model M_j is operating is ϖ_j. When it is, X has density function $p_j(x \mid \theta_j)$, where $\theta_j \in \mathbb{R}^{d_j}$ is a parameter having prior density $\pi_j(\theta_j)$ with respect to Lebesgue measure on $\mathbb{R}^{d_j}$. A *reversible jump Markov chain Monte Carlo (RJ-MCMC)* algorithm is used to estimate the posterior distribution over models, as well as within models, given data $X = x$. *For the special case that a move is proposed between distinct models of equal dimension*, explain how this operates, and in particular specify the acceptance probability. [No proofs are required.]

Random variables X_1, X_2 have independent Poisson distributions. There are two possible models. Under model M_1, X_i has mean γ_i $(i = 1, 2)$, where γ_1 and γ_2 are assigned independent $\Gamma(a, b)$ prior distributions ($a, b > 0$ known). Under model M_2, both X_1 and X_2 have the same mean, γ_1, with prior distribution $\Gamma(a, b)$. Initially both models are regarded as equally probable. We observe $(X_1, X_2) = (x_1, x_2)$ and wish to use a RJ-MCMC procedure to estimate the posterior probabilities of the two models. Describe suitable moves between the two models.

[*Hint: Regard M_2 as having an additional parameter γ_2 that has no effect on the distribution of* (X_1, X_2).]

END OF PAPER