UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 9:00 am to 11:00 am

PAPER 28

ACTUARIAL STATISTICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let N be the number of claims on a particular portfolio of insurance policies in a year and let p_n denote $\mathbb{P}(N = n)$, $n = 0, 1, 2, \ldots$.

(a) Suppose that

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \quad n = 1, 2, \ldots, \tag{1}$$

for some constants a and b. Show that the probability generating function $G_N(z)$ of N satisfies

$$(1 - az)G'_N(z) = (a + b)G_N(z).$$

(b) Now suppose that λ is a positive random variable with moment generating function M_λ, and that, given λ, N has a Poisson distribution with mean λ.

(i) Show that $G_N(z) = M_\lambda(z - 1)$.

(ii) Suppose that the random variable λ has the same distribution as $\alpha + Y$ where α is a positive constant and Y has an exponential distribution with mean $1/\beta$. Find an expression for G_N in terms of α and β.

Hence show that N has the same distribution as the sum of two independent random variables V and W, where V has a Poisson distribution and W has a geometric distribution. Use this to write down an expression for p_n as a finite sum.

Show that

$$(1 + \beta - z)G'_N(z) = \big\{1 + \alpha(\beta + 1) - \alpha z\big\}G_N(z).$$

Hence obtain an expression for p_n in terms of p_{n-1}, p_{n-2}, n, α and β, for $n \geqslant 2$. Explain how the p_ns may be calculated recursively. Show that when $\alpha = 0$, the p_ns satisfy (1) for some constants a and b which you should specify.

2

Let S be the total amount claimed in one year on a portfolio where the number N of claims in one year has a Poisson distribution with mean λ and the claim sizes are independent identically distributed positive random variables independent of N. Derive expressions for the mean and variance of S in terms of λ and the moments of the claim sizes. Find the moment generating function of S in terms of λ and the moment generating function of the claim sizes.

A portfolio contains two independent risks. The number of claims in one year on risk i has a Poisson distribution with mean λ_i, $i = 1, 2$. The claim sizes on risk i are independent identically distributed positive random variables with probability density function $f_i(x)$, mean μ_i and finite variance σ_i^2, $i = 1, 2$. For each risk, the claim sizes are independent of the number of claims. Let T be the total amount claimed in a year on the whole portfolio. Show that T has a compound Poisson distribution, and find the mean and variance of T.

Suppose that the direct insurer takes out a reinsurance contract on the portfolio, so that, for either risk, for a claim size x the direct insurer pays $g(x)$ and the reinsurer pays $x - g(x)$ for a given function g. Let T_I be the total amount paid by the direct insurer in one year, taking reinsurance into account. What is the distribution of T_I? Write down the function g, and the mean and variance of T_I,

(a) under quota share reinsurance with retained proportion α $(0 < \alpha < 1)$;

(b) under excess of loss reinsurance with retention level M (> 0).

3

In a classical risk model for a portfolio of insurance policies, claims arrive in a Poisson process with rate λ (> 0) and claims are independent identically distributed positive random variables with distribution function F, moment generating function M and mean μ, independent of the arrivals process. The premium rate is c (> 0). Assume that the relative safety loading ρ is positive and that there exists r_∞, $0 < r_\infty \leqslant \infty$, such that $M(r) \uparrow \infty$ as $r \uparrow r_\infty$. Define the probability of ruin $\psi(u)$ with initial capital u and write down an equation satisfied by the adjustment coefficient R.

Let $\varphi(u) = 1 - \psi(u)$. You are given that $\varphi(u)$ satisfies

$$\varphi(u) = 1 - \frac{\lambda\mu}{c} + \frac{\lambda}{c}\int_0^u \varphi(u-x)\{1 - F(x)\}dx.$$

Derive a renewal-type equation for $Z(u) = e^{Ru}\psi(u)$. Quoting without proof results from renewal theory as necessary, derive an expression for $\lim_{u\to\infty} Z(u)$.

Suppose that each claim is equally likely to be either exponentially distributed with mean 1 or exponentially distributed with mean 2. Show that

$$1 - F(x) = \frac{1}{2}\left(e^{-x} + e^{-x/2}\right),$$

and write down the value of μ. Explain why the adjustment coefficient R is in $(0, 1/2)$. Find $\lim_{u\to\infty} e^{Ru}\psi(u)$ in terms of ρ and R.

4

(a) Let X_i be the number of claims on a risk during accounting period i, $i = 1, 2, \ldots$, and suppose that, given λ, the X_is are conditionally independent and identically distributed. Suppose that $X_1, \ldots, X_n$ are observed. Derive the Bühlmann credibility estimate of $\mathbb{E}(X_{n+1}|\lambda)$.

(b) Suppose that, given λ, the X_is have a Poisson distribution with mean λ, and that *a priori* λ has a uniform distribution on $(1, 3)$.

(i) Suppose that one claim is observed in the first year. Find the Bühlmann credibility estimate of the expected number of claims in year 2.

(ii) Write down the Bühlmann credibility estimate of $\mathbb{E}(X_{n+1}|\lambda)$ based on observations $X_1 = x_1, \ldots, X_n = x_n$.

(iii) Suppose that x_{n+1} claims are observed in year $n+1$. For which values of x_{n+1} is the Bühlmann credibility estimate of $\mathbb{E}(X_{n+2}|\lambda)$ smaller that the Bühlmann credibility estimate in (ii)?

(iv) For the situation in (i), find the Bayesian estimate (under quadratic loss) of the expected number of claims in year 2.

END OF PAPER