UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 7 June, 2013 1:30 pm to 3:30 pm

PAPER 27

SCHRAMM–LOEWNER EVOLUTIONS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let K be a compact $\mathbb{H}$-hull and let $H = \mathbb{H} \setminus K$. Let $S \subset \delta H$, where δH stands for the intrinsic boundary $\hat{H} \setminus H$. Let $z = x + iy$ and let B be a complex Brownian motion.

(i) Let $z \in H$, and assume that $B_0 = z$. Let $T(H) = \inf\{t \geqslant 0 : B_t \notin H\}$. Show that as $t \uparrow T(H)$, B_t converges in $\hat{H}$ to a point $\hat{B}_{T(H)}$ of δH, almost surely. Show that for all Borel sets $S \subset \delta H$,

$$\lim_{y\to\infty, x/y\to 0} \pi y \mathbb{P}_z(\hat{B}_{T(H)} \in S) = \text{Leb } (g_K(S)),$$

where g_K is the mapping-out function of K. You may use without proof the following formula for the harmonic measure in $\mathbb{H}$:

$$h_{\mathbb{H}}(u + iv; t) = \frac{v}{\pi((t-u)^2 + v^2)}; \quad t \in \mathbb{R}.$$

(ii) Let $H_0 = \{x \in \mathbb{R} : H \text{ is a neighbourhood of } x \text{ in } \mathbb{H}\}$. By considering $S = \delta H \setminus H_0$ or otherwise, deduce briefly that

$$\text{cap}(K) := \lim_{y\to\infty} \pi y \mathbb{P}_{iy}(B_{T(H)} \in K)$$

is well-defined. You may assume the fact that g_K extends to a homeomorphism on $H \cup H_0$. Show that if K' is another compact $\mathbb{H}$-hull such that $K \subset K'$ then $\text{cap}(K) \leqslant \text{cap}(K')$. Deduce that $\text{cap}(K) \leqslant 4\,\text{rad}(K)$. [*Hint: If* $K' = \bar{\mathbb{D}} \cap \mathbb{H}$ *then* $g_{K'}(z) = z + z^{-1}$.]

(iii) Now assume that $\text{rad}(K) = 1$, that $0 \in \bar{K}$ and also that $x + iy \in \bar{K}$ with $x^2 + y^2 = 1$. By considering a reflection about the segment $I = [x, x + iy]$ show that $\text{cap}(K) \geqslant 1/2$. [*Hint: If* $K = [0, i]$ *then* $g_K(z) = \sqrt{z^2 + 1}$. *Consider the image of a segment* $(0, a]$ *under* g_K.]

2

Let $(\gamma_t, t \geqslant 0)$ be an $SLE(\kappa)$ curve, with driving function $(\xi_t, t \geqslant 0)$. Let $(g_t(z), t < \zeta(z), z \in \mathbb{H})$ denote the associated Loewner flow.

(i) For $z \in \mathbb{H}$, and $t < \zeta(z)$ let $h_t(z) = \arg(g_t(z) - \xi_t)$. Show that $(h_t(z), t < \zeta(z))$ is a martingale if and only if $\kappa = 4$.

(ii) Let $t > 0$ and let $H_t = \mathbb{H} \setminus \gamma[0, t]$. Show that $h_t(z)$ solves a Dirichlet problem in H_t and identify the boundary conditions.

(iii) Assume $\kappa = 4$. Let $z \in \mathbb{H} \setminus \gamma[0, \infty)$. Show that $F(z) = \lim_{t\to\infty} h_t(z)$ exists almost surely and compute $F(z)$.

3

(i) Write down Loewner's equation in the complex plane $\mathbb{C}$ for a driving function $(\xi_t, t \geqslant 0)$.

(ii) Let $\xi_t = -\sqrt{\kappa} W_t$, where W_t is a one-dimensional standard Brownian motion, and let g_t denote the associated Loewner flow. Let $x \in \mathbb{R}$ and let $X_t(x) = (g_t(x\sqrt{\kappa}) - \xi_t)/\sqrt{\kappa}$. Write down the stochastic differential equation for $X_t(x)$ (let $a = 2/\kappa$). Show that $X_t(x)$ hits zero a.s. if $a < 1/2$ and doesn't hit zero a.s. if $a > 1/2$.

(iii) For what range of values of the $SLE(\kappa)$ curve simple? Justify your answer.

4

(i) Define the notion of *compact* $\mathbb{H}$-*hull*. Let K be a compact $\mathbb{H}$-hull. Define the *mapping-out function* g_K and the *half-plane capacity* $\mathrm{hcap}(K)$. State (without proof) the *continuity* estimate and the *differentiability* estimate for g_K.

(ii) Let $(K_t)_{t \geqslant 0}$ be an increasing family of compact $\mathbb{H}$-hulls. Explain what it means to say that $(K_t)_{t \geqslant 0}$ satisfies the *local growth* property. Assuming the local growth property, define the associated *Loewner transform* $(\xi_t, t \geqslant 0)$, and show that $(\xi_t, t \geqslant 0)$ is continuous.

Assume that $\mathrm{hcap}(K_t) = 2t$. Find a differential equation satisfied by $g_t(z) = g_{K_t}(z)$, justifying your answer.

END OF PAPER