UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 9:00 am to 11:00 am

PAPER 26

PERCOLATION AND RELATED TOPICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $G = (V, E)$ be an arbitrary infinite, connected graph, with maximum vertex-degree $\Delta < \infty$. Define the term *self-avoiding walk* on G. The *length* π of a self-avoiding walk π is the number of its edges.

Let $\Sigma(v)$ be the set of all finite self-avoiding walks from vertex v. Let $\sigma_n(v)$ be the number of such self-avoiding walks of length n, and let $\sigma_n = \sup_{v \in V} \sigma_n(v)$. Show the existence of the limit

$$\mu := \lim_{n\to\infty} \sigma_n^{1/n}.$$

[You may appeal to any general result so long as a clear statement is given.]

(ii) (continuation) Suppose, in addition, that G has the property that $\sigma_n(u) = \sigma_n(v)$ for $n \geqslant 0$ and $u, v \in V$. Show that the power series

$$Z_G(x) = \sum_{\pi \in \Sigma(v)} x^{|\pi|}$$

has radius of convergence $1/\mu$. [You may assume that $x \in \mathbb{R}$ and $x > 0$.]

(iii) (continuation) Let G be, in addition, a bipartite graph. That is, the vertex-set V may be partitioned as $V_0 \cup V_1$ in such a way that every edge connects a vertex in V_0 to a vertex in V_1. Let $v \in V_0$, and let $\Sigma^0(v)$ be the subset of $\Sigma(v)$ containing all walks that end at vertices in V_0, and

$$Z_G^0(x) = \sum_{\pi \in \Sigma^0(v)} x^{|\pi|}.$$

Show that $Z_G^0(x) \leqslant Z_G(x) \leqslant (1 + \Delta x) Z_G^0(x)$.

(iv) (continuation) Suppose, in addition, that every vertex in V_1 has degree 3. Let the graph H be obtained from G by replacing every vertex $w \in V_1$ by a triangle, as illustrated in Figure 1, and let Z_H^0 be the generating function of self-avoiding walks from $v \in V_0$ on H that end in V_0.

Show that $Z_H^0(x) = Z_G^0(\sqrt{x^3 + x^4})$, and deduce that the radius ρ of convergence of Z_H^0 satisfies $\rho^3 + \rho^4 = \mu^{-2}$

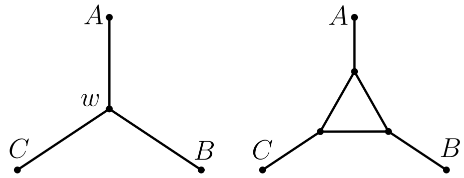

Figure 1: A vertex $w \in V_1$ is replaced by a triangle.

2

Let E be a finite set, and let P be a probability measure on the space $\Omega = \{0,1\}^E$. Give a clear statement of the FKG inequality for P. Show that a product measure on Ω satisfies the hypothesis of the FKG inequality.

Let $\mathbb{L}^d$ be the cubic lattice in $d \geqslant 2$ dimensions. Define the *critical probability* $p_c(d)$ of bond percolation on $\mathbb{L}^d$, and the *connective constant* $\mu(d)$ of $\mathbb{L}^d$.

Show that

$$p_c(d) \leqslant 1 - \frac{1}{\mu(2)}.$$

3

(a) Let (α_n) be a real sequence such that $\alpha_n/n \to 0$ as $n \to \infty$. Let the reals x_n satisfy $x_{m+n} \leqslant x_m + x_n + \alpha_n$ for $m, n \geqslant 1$. Show that the limit $\gamma = \lim_{n\to\infty} x_n/n$ exists and satisfies $\gamma \in [-\infty, \infty)$.

(b) Consider bond percolation on the d-dimensional cubic lattice $\mathbb{L}^d$ with $d \geqslant 2$ and edge-density $p \in (0,1)$. Let $\Lambda_n = [-n,n]^d$ and $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$. Give a careful proof that $\beta_n = \mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n)$ satisfies

$$\beta_{m+n} \leqslant |\partial\Lambda_n|\beta_m\beta_n, \qquad m, n \geqslant 1.$$

Deduce the existence of the limit $\lambda = -\lim_{n\to\infty} n^{-1}\log\beta_n$. [All logarithms are to base e.]

(c) Let $R(x) = \max\{k : x \leftrightarrow x + \partial\Lambda_k\}$ be the *radius* of the open cluster at x, and let $M_n = \max\{R(x) : x \in \Lambda_n\}$ be the maximum radius of open clusters at points in Λ_n. Assume that p is such that $\lambda > 0$. Show that, for $\epsilon > 0$,

$$\mathbb{P}_p\left(1 - \epsilon < \frac{M_n}{(d/\lambda)\log n} < 1 + \epsilon\right) \to 1 \quad \text{as } n \to \infty.$$

4

Write an essay on the phase transition in the percolation model. Your essay should contain clear definitions of the terms in use, and should include discussions of critical exponents, universality, and the relevance of the number d of dimensions. Special attention should be given to the case $d = 2$. Your exposition should concentrate more on the communication of ideas than on giving complete proofs.

END OF PAPER