UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 9:00 am to 12:00 pm

PAPER 25

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

(a) Show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a process $X = \{X(h) : h \in H\}$ can be defined with the properties that

1. $X(ag + bh) = aX(g) + bX(h)$ almost surely for all $a, b \in \mathbb{R}$ and $g, h \in H$, and
2. $X(h) \sim N(0, \|h\|^2)$ for all $h \in H$.

[You may use, without proof, the following facts: there exists a probability space on which an i.i.d. sequence of $N(0, 1)$ random variables can be defined, and there exists a countable orthonormal basis of H.]

(b) Show that the process X from part (a) is such that the random variables $X(h_1), \ldots, X(h_n)$ are jointly normal for all $h_1, \ldots, h_n \in H$. What is $\mathrm{Cov}(X(g), X(h))$?

(c) Without appealing to Wiener's theorem on the existence of Brownian motion, prove that there exists a probability space on which a process $(W_t)_{t \geqslant 0}$ can be defined with the properties that

1. $W_0 = 0$,
2. $W_t - W_s \sim N(0, t - s)$ for all $0 \leqslant s < t$, and
3. the random variables $W_{t_0} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent for all $0 \leqslant t_0 < t_1 < \cdots < t_n$.

(d) Let X be the process from part (a). Fix $k \in H$ and define an equivalent probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{X(k) - \|k\|^2/2}.$$

Show that, under this measure $\mathbb{Q}$, the random variable $X(h) - \langle h, k \rangle$ has the $N(0, \|h\|^2)$ distribution.

2

Let $(W_t)_{t\geqslant 0}$ be a real Brownian motion generating the right-continuous, completed filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Let $T > 0$ be a non-random constant and let ϕ be a bounded, continuously differentiable function with bounded derivative ϕ'.

(a) Show that for any $0 \leqslant s < t$ and bounded $\mathcal{F}_s$-measurable random variable K we have

$$\mathbb{E}\left[\phi\left(W_T\right) K(W_t - W_s)\right] = \mathbb{E}\left[\phi'\left(W_T\right) K(T \wedge t - T \wedge s)\right]$$

[You may use the following fact without proof: if X and Y are jointly normal random variables with $\mathbb{E}(Y) = 0$, then $\mathbb{E}[\phi(X)Y] = \mathbb{E}[\phi'(X)]\mathrm{Cov}(X, Y)$.]

(b) Show that for any predictable α such that $\mathbb{E}\left(\int_0^\infty \alpha_u^2 du\right) < \infty$ we have

$$\mathbb{E}\left[\phi\left(W_T\right) \int_0^\infty \alpha_u \, dW_u\right] = \mathbb{E}\left[\int_0^T \phi'\left(W_T\right) \alpha_u \, du\right].$$

(c) Use Itô's martingale representation theorem to prove that there exists a predictable process β such that $\mathbb{E}\left(\int_0^\infty \beta_u^2 du\right) < \infty$ and a constant c such that

$$\phi\left(W_T\right) = c + \int_0^\infty \beta_u \, dW_u$$

How is c related to $\phi(W_T)$? In what sense is the process β unique? You may use standard facts about martingales without proof.

(d) Use part (b) to show that

$$\mathbb{E}\left[\int_0^\infty \left(\beta_u - \phi'(W_T)\mathbb{1}_{\{t\leqslant T\}}\right) \alpha_u \, du\right] = 0$$

for any predictable α such that $\mathbb{E}\left(\int_0^\infty \alpha_u^2 du\right) < \infty$.

(e) Conclude that

$$\beta_t = \mathbb{E}\left[\phi'\left(W_T\right) | \mathcal{F}_t\right] \mathbb{1}_{\{t\leqslant T\}}$$

up to the sense of uniqueness from part (c).

3

Let $(X_t)_{0\leqslant t\leqslant 1}$ be a continuous martingale with respect to a filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant 1}$. Suppose that $X_0 = 0$ and that there exists constant $C > 0$ such that $|X_t(\omega)| \leqslant C$ for all $(t,\omega) \in [0,1]\times\Omega$. Let

$$A_t^{(n)} = \sum_{k=1}^{2^n} (X_{k2^{-n}\wedge t} - X_{(k-1)2^{-n}\wedge t})^2$$

and

$$M_t^{(n)} = \frac{1}{2}(X_t^2 - A_t^{(n)}).$$

(a) Show that

$$\mathbb{E}[(M_1^{(n)})^2] \leqslant C^4$$

(b) Show that

$$\mathbb{E}[(A_1^{(n)})^2] \leqslant 10C^4$$

(c) Using the formula

$$M_1^{(n)} - M_1^{(m)} = \sum_{j=1}^{2^n-1} (X_{j2^{-n}} - X_{\lfloor j2^{m-n}\rfloor 2^{-m}})(X_{(j+1)2^{-n}} - X_{j2^{-n}}),$$

for $n > m$, show that

$$\mathbb{E}[(M_1^{(n)} - M_1^{(m)})^2] \to 0 \text{ as } m,n \to \infty.$$

Here the symbol $\lfloor y\rfloor$ denotes the greatest integer less than or equal to y.

(d) Show that

$$\mathbb{E}[\sup_{0\leqslant t\leqslant 1} (A_t^{(n)} - A_t^{(m)})^2] \to 0 \text{ as } m,n \to \infty.$$

You may use, without proof, the fact that $M^{(n)}$ is a continuous martingale for each n.

4

Consider the SDE

$$dX_t = \tanh X_t \; dt + dW_t \qquad (*)$$

where W is a scalar Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(a) Prove that (*) has a unique strong solution starting from any intial point X_0. You may use, without proof, any theorem from the course as long as it is clearly stated.

(b) Show that the process $\frac{e^{t/2}}{\cosh X_t}$ is a postitive martingale.

[Recall that $\frac{d}{dx}\cosh x = \sinh x$ and $\tanh x = \frac{\sinh x}{\cosh x}$.]

(c) Fix constants $T > 0$ and $X_0 = x$, define an equivalent probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{\cosh x}{\cosh X_T} e^{T/2}$$

Show that the process $(\hat{W}_t)_{0 \leqslant t \leqslant T}$ is a $\mathbb{Q}$-Brownian motion, where $\hat{W}_t = X_t - x$.

(d) Use part (c) to compute the density of the random variable X_T under the measure $\mathbb{P}$.

5

Let X be a continuous local martingale in a filtration $(\mathcal{F}_t)_{t \geqslant 0}$, and suppose that $X_0 = 0$ and that X has conditionally symmetric increments in the sense that

$$\mathbb{E}[g(X_t - X_s)|\mathcal{F}_s] = \mathbb{E}[g(X_s - X_t)|\mathcal{F}_s]$$

for all $0 \leqslant s \leqslant t$ and bounded measurable g. Suppose that the filtration is such that each martingale has a continuous modification.

Fix constants $\theta \in \mathbb{R}$ and $T > 0$ and let M be the continuous martingale such that

$$M_t = \mathbb{E}(e^{\mathrm{i}\theta X_T}|\mathcal{F}_t) \text{ for } 0 \leqslant t \leqslant T,$$

where $\mathrm{i} = \sqrt{-1}$.

(a) Show that the process $e^{-\mathrm{i}2\theta X_t} M_t$ is a martingale.

(b) Show that $d\langle M, X\rangle_t = \mathrm{i}\theta M_t d\langle X\rangle_t$.

(c) Show that the process $e^{-\mathrm{i}\theta X_t - \frac{1}{2}\theta^2 \langle X\rangle_t} M_t$ is a martingale.

(d) Conclude that $\mathbb{E}(e^{\mathrm{i}\theta X_T}) = \mathbb{E}(e^{-\frac{1}{2}\theta^2\langle X\rangle_T})$.

(e) Suppose $X_t \sim N(0, t)$ for all $t \geqslant 0$. Show that X is a Brownian motion.

6

Let X be a d-dimensional weak solution of the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ are given continuous functions and W is an n-dimensional Brownian motion. Assume that the initial condition $X_0 \in \mathbb{R}^d$ is not random.

(a) Show that there exists a second order partial differential operator $\mathcal{L}$ such that

$$f(X_t) - \int_0^t (\mathcal{L}f)(X_s)ds$$

is a local martingale for all twice-continuously differentiable functions f.

(b) Let $\lambda > 0$ be a given constant, and let $u : \mathbb{R}^d \to \mathbb{R}$ be a given twice-continuously differentiable function such that

$$\mathcal{L}u = \lambda u$$

Show that the process M defined by $M_t = e^{-\lambda t}u(X_t)$ is a local martingale.

(c) Let $\mathcal{D} \subset \mathbb{R}^d$ be an open set. Define a stopping time by

$$T = \inf\{t \geqslant 0 : X_t \in \partial\mathcal{D}\}$$

where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$. Suppose u is bounded on $\mathcal{D} \cup \partial\mathcal{D}$. Assuming $X_0 \in \mathcal{D}$, show that the stopped process $M^T = (M_{t\wedge T})_{t\geqslant 0}$ is a martingale, and converges a.s. as $t \to \infty$.

(d) Suppose further that $u(x) = 1$ for all $x \in \partial\mathcal{D}$. Show that

$$\mathbb{E}(e^{-\lambda T}) = u(X_0)$$

(e) Let B be a real Brownian motion and $a \in \mathbb{R}$ a constant, and define a stopping time by

$$S = \inf\{s \geqslant 0 : y + B_s + as = 0\}$$

where $y > 0$ is a constant. For every $\lambda > 0$, compute

$$\mathbb{E}(e^{-\lambda S})$$

in terms of a, λ and y.

END OF PAPER