UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 9:00 am to 12:00 pm

PAPER 24

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) State the almost sure martingale convergence theorem.
Give an example of a martingale that satisfies the hypotheses of the almost sure martingale convergence theorem, but fails to converge in L^1.

b) Let $X = (X_n : n \geqslant 1)$ be a zero mean martingale in L^2. Show that, for $\lambda > 0$,

$$\mathbb{P}\left(\max_{1 \leqslant k \leqslant n} X_k \geqslant \lambda\right) \leqslant \frac{\mathbb{E}[X_n^2]}{\lambda^2 + \mathbb{E}[X_n^2]}.$$

[*Hint: The function* $x \mapsto (x+c)^2$ *is convex.*]

2

a) State the definition of a Lévy process.

b) Let φ_X denote the characteristic function of a random variable X. Prove from first principles that the map

$$t \mapsto \varphi_{X_t}(u)$$

is continuous for every $u \in \mathbb{R}$ if $X = (X_t : t \geqslant 0)$ is a Lévy process.

c) Again, from first principles, show that if X is a Lévy process, then

$$\varphi_{X_t}(u) = e^{t\eta(u)}$$

for some function $\eta = \eta(u)$.

[*Hint: You may use the fact that, for a continuous function* $f : \mathbb{R} \to \mathbb{C}$, *the requirements* $f(0) = 1$ *and* $f(s+t) = f(t)f(s)$ *uniquely characterize the exponential function.*]

d) Construct a Lévy process X that has

$$\varphi_{X_t}(u) = e^{2t(\cos u - 1)}.$$

Describe the sample paths of this process.

3

Let $B = (B_t \colon t \geqslant 0)$ be a standard Brownian motion on $\mathbb{R}$.

a) Determine functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ such that the processes

$$X = (B_t^3 + \alpha(t)B_t \colon t \geqslant 0)$$

and

$$Y = (B_t^4 + \beta(t)B_t^2 + \gamma(t) \colon t \geqslant 0)$$

are martingales.

b) For $x > 0$ and $B = (B_t \colon t \geqslant 0)$ a standard Brownian motion on $\mathbb{R}$, set

$$\tau_x = \inf\{t \geqslant 0 \colon B_t \notin (-x, x)\}.$$

Compute

$$\mathbb{E}[\tau_x] \quad \text{and} \quad \mathbb{E}[\tau_x^2].$$

Also, for $\lambda > 0$, determine the function

$$\mathbb{E}[e^{-\lambda \tau_x}].$$

4

Let $X = (X_t \colon t \geqslant 0)$ and $Y = (Y_t \colon t \geqslant 0)$ be stochastic processes in continuous time.

a) What does it mean to say that Y is a version of X? What does it mean to say that X and Y are indistinguishable?

Suppose Y is a version of the process X. Show by example that X and Y need not be indistinguishable.

b) Suppose Y is a version of X. Prove that if X and Y have the additional property that their sample paths are cadlag almost surely, then X and Y are indistinguishable.

c) State the definition of a progressively measurable process. Prove that if X is a cadlag adapted process, then X is progressively measurable.

d) Let X be a cadlag adapted process, and let τ be a stopping time. Prove that the stopped process X^τ is adapted.

5

Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables with zero mean and unit variance, and define

$$S_n = \sum_{k=1}^{n} X_k, \quad n = 1, 2, 3, \ldots$$

a) Formulate Skorokhod's embedding theorem for random walks, as well as Donsker's invariance principle.

b) Prove that the law of

$$\frac{1}{n^{3/2}} \sum_{k=1}^{n} S_k$$

converges weakly to that of the random variable

$$\int_0^1 B_t dt,$$

where $B = (B_t \colon t \geqslant 0)$ is a standard Brownian motion on $\mathbb{R}$.
[*Hint: The sums* $\sum_{k=1}^{n} S_k/n^{3/2}$ *approximate a certain continuous operation on the space* $C[0,1]$.]

6

Let $B = (B_t : t \geqslant 0)$ be a standard Brownian motion.

a) Show that, almost surely,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{t}} = \infty.$$

b) Prove that, almost surely,

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} \leqslant 1.$$

[*Hint: Set* $\Phi(t) = (2t\log\log t)^{1/2}$ *and consider events of the form*

$$\left\{ \sup_{0\leqslant t\leqslant a^n} B_t \geqslant (1+\epsilon)\Phi(a^n) \right\}.$$

Note that $\Phi(t)/t$ *is a decreasing function. You may find the tail bound*

$$\mathbb{P}(X > x) \leqslant e^{-x^2/2}$$

for $\mathcal{N}(0,1)$*-random variables useful.*]

END OF PAPER