UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 9:00 am to 12:00 pm

PAPER 23

TOPICS IN ANALYTIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Explain what it means for a Dirichlet character to be primitive. Let $\chi \pmod q$ be a non-principal character. Define, in terms of χ, the primitive character inducing χ. Why is this unique?

(b) Assume that q is a power of a prime. Define the Fourier transform $\hat{f}$ of a function f in $L^2(\mathbb{Z}/q\mathbb{Z})$. Prove that if χ is any Dirichlet character modulo q and $(a,q)=1$ then

$$\hat{\chi}(a) = \overline{\chi}(a)\hat{\chi}(1). \tag{1}$$

Prove that if χ is a primitive character then (1) holds even if $(a,q)>1$.

(c) Continue to assume that q is a power of a prime. Show that for any character $\chi \pmod q$, $|\hat{\chi}(1)| \leqslant 1$, with equality if χ is primitive.

(d) Let q be an odd prime and let $\chi \pmod q$ be a non-principal Dirichlet character. Prove the Pólya-Vinogradov inequality: for any $M, N \in \mathbb{Z}$ with $N \geqslant 1$,

$$\left| \sum_{M<n\leqslant M+N} \chi(n) \right| \ll \sqrt{q}\, \log q.$$

You may assume, without proof, that for integers $q \neq 0$ and a, with q not dividing a, we have the bound

$$\left| \sum_{M<n\leqslant M+N} e\left(\frac{a}{q}\right) \right| \ll \left\| \frac{a}{q} \right\|_{\mathbb{R}/\mathbb{Z}}^{-1},$$

where $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ is the usual norm on $\mathbb{R}/\mathbb{Z}$.

(e) Continue to assume that q is an odd prime. We say that $a \pmod q$ is a square if there exists $x \pmod q$ such that $x^2 \equiv a \pmod q$. Using part (d) or otherwise, show that the number of squares modulo q in an interval $(M, M+N]$ is

$$\frac{N(q+1)}{2q} + O(\sqrt{q}\log q).$$

In answering this question you may quote the Plancherel formula without proof.

2

(a) What does it mean for a Dirichlet character to be even or odd? Write down the θ-function θ_χ associated to primitive even and primitive odd Dirichlet characters modulo q and state the automorphy relationship between $\theta_\chi(x)$ and $\theta_\chi(\frac{1}{x})$.

(b) Given a primitive character $\chi \pmod q$ write down its completed L-function $\Lambda(s,\chi)$. Assuming the automorphy relation for the related θ-function, prove the functional equation of $\Lambda(s,\chi)$ in the case where χ is even. Describe any differences between the zero set of $L(s,\chi)$ and that of $\Lambda(s,\chi)$ when χ is either even or odd.

(c) Now suppose that $\chi \bmod q$ is an imprimitive even character, induced by primitive $\chi' \bmod q'$. Write down a relation between $L(s,\chi)$ and $L(1-s,\overline{\chi})$ in terms of χ'. Describe the zero set of $L(s,\chi)$ in terms of the zero set of $L(s,\chi')$. Under what conditions are the two sets the same?

(d) Let $t \geqslant 2$ and let $\chi \bmod q$ be a primitive even character. Give the proof that

$$|L(it,\chi)| \ll t^{1/2} q^{1/2} \log(q+t).$$

In answering part (d) you may assume any facts proven in lectures regarding the completed L-function $\Lambda(s,\chi)$ so long as you clearly state what you are using. You may also assume the following consequence of Stirling's approximation:

There exist $c_1 > c_2 > 0$ such that for $0 \leqslant \sigma \leqslant 2$ and $t \geqslant 2$

$$c_2 e^{\frac{-\pi t}{2}} t^{\sigma-1/2} \leqslant |\Gamma(\sigma+it)| \leqslant c_1 e^{\frac{-\pi t}{2}} t^{\sigma-1/2}.$$

3

(a) Write down the definition of $\xi(s)$, the completed Riemann ζ function, and write down its Hadamard product. Prove, assuming Jensen's formula, that the number of zeros of $\xi(s)$ in a ball of radius $T > 2$ about 0 is $O(T \log T)$.

(b) When $\xi(\sigma+it) \neq 0$, the real part of the logarithmic derivative of ξ may be expressed as

$$\Re\left(\frac{\xi'}{\xi}(\sigma+it)\right) = \sum_{\substack{\xi(\rho)=0\\ \rho=\beta+i\gamma}} \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2},$$

a fact which you may assume without proof. Prove that this is a convergent sum.

(c) Let t be large. Show that

$$\sum_{\substack{\xi(\rho)=0\\ \rho=\beta+i\gamma}} \frac{1}{1+|t-\gamma|^2} = O(\log t).$$

Hence deduce that for large T, ζ has $O(\log T)$ zeros with imaginary part in the interval $[T, T+1]$.

(d) Prove that the Riemann Hypothesis is equivalent to the statement that, for each fixed $t \in \mathbb{R}$ and for $\sigma \geqslant 1/2$, $|\xi(\sigma+it)|$ is an increasing function of σ.

In your proofs you may assume the following two consequences of Stirling's approximation.

i. For each $\epsilon > 0$, uniformly for $-\pi+\epsilon < \arg(z) < \pi-\epsilon$, we have

$$\log\Gamma(z) = z\log z - z + O(1+|z|^{-1}).$$

ii. For $\Re(z) > 0$,

$$\frac{\Gamma'}{\Gamma}(z) = \log z + O\left(\frac{1}{|z|}\right).$$

4

(a) Let $L(s,\chi)$ be the L-function associated to a complex primitive Dirichlet character $\chi \bmod q$. Either by considering a product containing $L(s,\chi)$ and other L-functions, or otherwise, prove that for all real t, $L(1+it,\chi) \neq 0$.

(b) Let χ_D given by $\chi_D(n) = \left(\frac{D}{n}\right)$ be a primitive real character of conductor $|D| > 1$. Prove that the function $\zeta(s)L(s,\chi_D) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ has $a_n \geqslant 0$.

(c) Using the function from part c, or otherwise, prove that there exists a constant $c > 0$ such that $L(s,\chi_D)$ does not have two real zeros in the interval $[1 - c/\log|D|, 1]$.

In answering this question, you may find it useful to quote the partial fraction representation of $\frac{-L'}{L}(s,\chi)$,

$$-\frac{L'}{L}(\sigma+it,\chi) = O(\log q + \log(1+|t|)) - \sum_{\substack{L(\rho,\chi)=0\\ \rho=\beta+i\gamma\\ \beta>0\\ |t-\gamma|<1}} \frac{1}{\sigma+it-\rho},$$

valid for $\sigma > 1/2$ *and for any non-principal* $\chi \bmod q$.

5

(a) Define the norm $\| \cdot \|_{\mathbb{R}/\mathbb{Z}}$ on $\mathbb{R}/\mathbb{Z}$. Let $\delta > 0$. Explain what it means for a set $\{x_1, ..., x_R\} \subset \mathbb{R}/\mathbb{Z}$ to be δ-well-spaced.

(b) Explain the principle of duality as it pertains to the analytic large sieve inequality. State one of the equivalent forms of the analytic large sieve and sketch the proof, giving at least some of the more interesting technical details. You may assume the Poisson summation formula and any properties of the Féjer kernel, so long as you clearly state what that you are using.

(c) Define the set of Farey fractions of level Q. Prove that this set is Q^{-2}-well-spaced.

(d) State the variant of the multiplicative large sieve inequality that was proven in lectures and used in the proof of Linnik's theorem. Using this, or otherwise, prove the following statements regarding primes in short intervals.

(i) Let $\pi(x)$ denote the number of primes less than x. Let $2 \leqslant N < M$. Prove

$$\pi(M+N) - \pi(M) \ll \frac{N}{\log N}.$$

(ii) More generally, let $q > 1$, $(a,q) = 1$ and let $\pi(x;q,a)$ denote the number of primes p less than x satisfying $p \equiv a \pmod{q}$. Let $\delta > 0$ and $q^{2+\delta} \leqslant N < M$. Prove

$$\pi(M+N;q,a) - \pi(M;q,a) \ll \frac{N}{\phi(q) \log N}.$$

6

(a) State the Riemann Hypothesis and the Lindelöf Hypothesis.

(b) Sketch the proof that the Riemann Hypothesis implies the Lindelöf Hypothesis, giving at least some of the more interesting technical details.

You may find the following pair of formulae useful

$$-\frac{\zeta'}{\zeta}(\sigma+it) = \sum_{p^n \leqslant x} \frac{\log p}{p^{n(\sigma+it)}} + \sum_{x < p^n \leqslant x^2} \frac{\log p}{p^{n(\sigma+it)}} \frac{\log \frac{x^2}{p^n}}{\log x}$$
$$+ \sum_{\xi(\rho)=0} \frac{x^{2(\rho-\sigma-it)} - x^{\rho-\sigma-it}}{(\rho-\sigma-it)^2 \log x} + O\left(\frac{x^{-\frac{1}{2}-\sigma}\log t}{\log x}\right),$$

valid for $\sigma > \frac{1}{2}$ *and* $2 < x < t$, *and*

$$\Re\frac{\xi'}{\xi}(\sigma+it) = \sum_{\xi(\frac{1}{2}+i\gamma)=0} \frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}$$

valid for all σ, t.

You may also assume that for $\Re(s) > 0$, $\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(\frac{1}{|s|}\right)$.

END OF PAPER