UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 9:00 am to 12:00 pm

PAPER 21

ALGEBRAIC NUMBER THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What is a *valuation* on a field K? Show that every valuation on $\mathbb{Q}$ is equivalent to the p-adic valuation v_p for some prime p.

State a form of Hensel's Lemma, and use it to determine the number of roots of each of the following polynomials:

$$X^3 + 1 \text{ in } \mathbb{Q}_7$$
$$X^2 + 2X + 4 \text{ in } \mathbb{Q}_2$$
$$3X^3 + X + 3 \text{ in } \mathbb{Q}_3$$

2

Let L/K be a finite extension of fields which are complete with respect to a discrete valuation. What does it mean to say that L/K is (i) *unramified* (ii) *totally ramified*?

Show that L/K is unramified if and only if $L = K(x)$ for some $x \in \mathfrak{o}_L$ for which the reduction $\overline{f_{x,L/K}}$ of its minimal polynomial is separable over the residue field of K.

Let L be a finite extension of $\mathbb{Q}_p$ and $f = f(L/\mathbb{Q}_p)$ its residue class degree. Show that L contains all $(p^f - 1)$-th roots of unity. Show that if $e(L/\mathbb{Q}_p) < p-1$ then L contains no other roots of unity.

3

(i) Define the *ramification groups* $G_i(L/K)$ for a Galois extension L/K of finite extensions of $\mathbb{Q}_p$. Show that if L/K is totally ramified and π_L is a uniformiser of L, then

$$G_i(L/K) = \{\sigma \in \mathrm{Gal}(L/K) \mid v_L(\sigma(\pi_L) - \pi_L) \geqslant i+1\} .$$

(ii) Compute the ramification groups of L/K in the following cases:

(a) $K = \mathbb{Q}_p$, $L = \mathbb{Q}_p(\zeta_{p^n})$ where ζ_{p^n} is a primitive p^n-th root of unity;

(b) $K = \mathbb{Q}_3$, $L = \mathbb{Q}_3(\zeta_3, \sqrt[3]{3})$.

4

What is a *place* of a number field K?

Show that there is a bijection between the set of places of K lying over p and the set of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-equivalence classes of embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$.

Prove the product formula: if $x \in K^*$ then $|x|_v = 1$ for all but finitely many places v of K and

$$\prod_v |x|_v = 1.$$

Show that only finitely many primes ramify in $K/\mathbb{Q}$.

5

Define the *idele group* J_K of a number field K. Describe the topology on J_K. Show that K^* embeds as a discrete subgroup of J_K.

Let $\mathfrak{m} = \sum m_v(v)$ be a modulus of K. Define the *ray class group* $Cl_{\mathfrak{m}}(K)$. Suppose $m_v = 0$ for every infinite place v of K. Show that there is an exact sequence

$$\mathfrak{o}_K^* \to \prod_{v \in S} \frac{\mathcal{O}_v^*}{1 + \pi_v^{m_v}\mathcal{O}_v} \to Cl_{\mathfrak{m}}(K) \to Cl(K) \to 0$$

where $S = \{v \mid m_v \neq 0\}$. Compute $Cl_{\mathfrak{m}}(\mathbb{Q}(\sqrt{2}))$ when $\mathfrak{m} = 2(v) + (v')$, where v, v' are the places given by the prime ideals $(\sqrt{2})$, $(1 + 2\sqrt{2})$ respectively. (You may assume that $1 + \sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$.)

END OF PAPER