UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 1:30 pm to 4:30 pm

PAPER 19

TOPICS IN SET THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Theorems and other propositions from lecture notes may be quoted without proof, provided that (1) the results are stated fully and precisely, and (2) their proof is not specifically requested.

The notation follows the course lecture notes. In particular, ZFC stands for the first-order theory whose axioms are the axioms of Zermelo Frankel set theory with the axiom of Choice, formulated in the vocabulary $\{\in\}$. For cardinals κ and λ, $[\kappa]^\lambda$ denotes the set of λ-element subsets of κ; $lim(\kappa)$ is the set of limit ordinals less than κ. The forcing convention is: $p \leqslant_{\mathbb{P}} q$ means that the condition q is stronger than the condition p in the partial order $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$.

1

(i) Define carefully any <u>two</u> of the following three italicized concepts:

(a) the class $\mathbb{A}$ is an *inner model* of the theory ZFC;

(b) the ordinal λ is a *strongly inaccessible cardinal*;

(c) the formula $\varphi(x_1, \ldots, x_n)$ in the first-order language of set theory is *absolute* for the classes $\mathbb{A}$ and $\mathbb{B}$.

(ii) (a) Suppose that $\{A_\alpha : \alpha \in Ord\}$ is a cumulative hierarchy for the class A. State and prove the Reflection Theorem for ZFC relative to A and $\{A_\alpha : \alpha \in Ord\}$.

(b) Deduce that if φ is a formula in the vocabulary of ZFC such that $ZFC \vdash \varphi$, then there exists an ordinal δ such that for some countable transitive elementary submodel $\mathbb{M} \subseteq V_\delta, \mathbb{M} \models \varphi$.

(iii) Suppose that there exists an ordinal ξ such that for every axiom ψ of ZFC, $V_\xi \models \psi$. Let $\xi^\star$ be the least such ordinal. Show that the cofinality of $\xi^\star$ is ω.

(iv) Suppose $\kappa = cf(\kappa)$. Prove that if the class H_κ of sets of cardinality hereditarily less than κ is a model of ZFC, then κ is a strongly inaccessible cardinal.

2

(i) State carefully any <u>two</u> of the following three italicized assertions:

(a) the *Axiom of Constructibility* $V = L$;

(b) the *Bukovský-Hechler Theorem* for a singular cardinal κ;

(c) the *Singular Cardinals Hypothesis.*

(ii) Suppose δ is a limit ordinal. Prove $cf(cf(\delta)) = cf(\delta)$.

(iii) Let κ be a limit cardinal and λ be a non-zero cardinal. Let δ be a limit ordinal such that $\lambda < cf(\delta)$. Suppose that $\{\kappa_\xi < \kappa : \xi < \delta\}$ is a strictly increasing sequence of cardinals such that $\kappa = \sum_{\xi<\delta} \kappa_\xi$. Show that $\kappa^\lambda = \sum_{\xi<\delta} {\kappa_\xi}^\lambda$.

(iv) (a) Suppose that $\kappa = cf(\kappa)$. Prove by induction on $\zeta < \kappa^+$ that there exist functions $S_\zeta : \kappa \to \kappa$ such that for all $\alpha < \beta$ there exists γ with the property that $(\forall\delta > \gamma)(S_\alpha(\delta) < S_\beta(\delta))$.

(b) Suppose that κ and λ are infinite cardinals. Let $*(\kappa, \lambda)$ denote the following assertion: there exists a family $\{S_\zeta : \zeta < \kappa^+\} \subseteq [\kappa]^\lambda$ such that for each $I \in [\kappa^+]^\kappa$, there exists $\{S^*_\zeta : \zeta \in I\}$, where $S^*_\zeta \subseteq S_\zeta$, $| S_\zeta \setminus S^*_\zeta | < \lambda$, and $S^*_\alpha \cap S^*_\beta = \emptyset$ if $\alpha \neq \beta$.
By considering the family $\{S_\zeta : \zeta < \kappa^+\}$ of subsets of $\kappa \times \kappa$ from the previous part or otherwise, prove that if κ is a regular cardinal, then $*(\kappa, \kappa)$ holds.

3

(i) Define carefully any <u>two</u> of the following three italicized expressions:

(a) the set $S \subseteq \delta$ is a *stationary subset* of the limit ordinal δ;

(b) the partial order $\mathbb{T} = (T, \leqslant_{\mathbb{T}})$ is a κ-*tree*, where κ is a cardinal;

(c) the *prediction principle* $\clubsuit$.

(ii) (a) Prove there exists a family $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ of sets such that

(1) for every $\alpha < \omega_1$, the set $\omega_1 \setminus \bigcup_{n<\omega} A_{\alpha,n}$ is countable;

(2) if $\alpha \neq \beta < \omega_1$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for all $n < \omega$.

(b) Suppose $\lambda = cf(\lambda) > \aleph_0$. Let $B : \lambda \to [\lambda]^{\aleph_0}$ be a function. Show that the set $E = \{\delta \in lim(\lambda) : (\forall \alpha < \delta)(B(\alpha) \subseteq \delta)\}$ is closed unbounded in λ.

(iii) Suppose that S is a stationary subset of $\lambda = cf(\lambda) > \aleph_0$. Prove that $\diamondsuit_S$ and $\diamondsuit'_S$ are equivalent in ZFC, where $\diamondsuit'_S$ is the statement: there exists $\{E_\alpha : \alpha \in S\}$ such that

(1) $\forall \alpha \in S, E_\alpha$ is a countable family of subsets of α;

(2) $\forall X \subseteq \lambda, \{\alpha \in S : X \cap \alpha \in E_\alpha\}$ is a stationary subset of λ.

(iv) Suppose that κ and λ are regular cardinals, $\kappa < \lambda$, and $\mathbb{T} = (T, \leqslant_{\mathbb{T}})$ is a λ-tree, each of whose levels has cardinality less than κ. Using Fodor's Lemma or otherwise, determine whether $\mathbb{T}$ has a cofinal branch. (It may help to assume that $T = \lambda$.)

4

(i) Suppose $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ is a forcing in a model $\mathbb{M}$ of ZFC, and $p \in \mathbb{P}$.
Define carefully any <u>two</u> of the following three italicized terms:

(a) the set τ is $\mathbb{P}$-*name*;

(b) the set G is a *generic filter* in $\mathbb{P}$ over $\mathbb{M}$;

(c) the set E is *dense above* p in $\mathbb{P}$.

(ii) Let G be a generic filter in a forcing $\mathbb{P}$ over a countable transitive model $\mathbb{M}$.

(a) Prove that $Ord \cap \mathbb{M} = Ord \cap \mathbb{M}[G]$.

(b) Formulating precisely any auxiliary results about forcing to which you appeal, show that $\mathbb{M}[G] \models \varphi$, where φ is an instance of the axiom schema of Separation.

(iii) Prove that a filter G is generic in a forcing $\mathbb{P}$ over a countable transitive model $\mathbb{M}$ if and only if for every anti-chain $A \in \mathbb{M}$ which is maximal in $\mathbb{P}$, $\mid G \cap A \mid = 1$.

(iv) Suppose that the forcing $\mathbb{P}$ has the $\aleph_2$-chain condition in the countable transitive model $\mathbb{M}$ and (S is a stationary subset of $\omega_2)^{\mathbb{M}}$. Let G be a generic filter in $\mathbb{P}$ over $\mathbb{M}$. Prove that (S is a stationary subset of $\omega_2)^{\mathbb{M}[G]}$.

5

(i) Let H be a generic filter in a forcing $\mathbb{Q}$ over a countable transitive model $\mathbb{M}$; let q be a condition in $\mathbb{Q}$. Suppose φ is a formula in the vocabulary of ZFC which may include $\mathbb{Q}$-names. Define or state carefully any two of the following three italicized concepts:

(a) the *value* $val(\tau, H)$ of the $\mathbb{Q}$-name τ;

(b) the set σ is a *nice* $\mathbb{Q}$-*name*;

(c) the *forcing relation* $q \Vdash_{\mathbb{Q}} \varphi$.

(ii) (a) Suppose that the partial order $\mathbb{T} = (T, \leqslant_{\mathbb{T}})$ is an $\aleph_1$-tree in which every anti-chain is countable. Show that $\mathbb{T}$ has no uncountable chains.

(b) Prove that $MA_{\aleph_1}$ implies Suslin's Hypothesis.

(iii) Show that the existence of a Suslin tree does not imply the Continuum Hypothesis.

(iv) Suppose that κ is a regular cardinal. Let $\Box_\kappa$ be the following assertion: there exists a sequence $\langle C_\zeta : \zeta \in lim(\kappa^+)\rangle$ such that

(1) the set $C_\zeta \subseteq \zeta$ is club in ζ;

(2) if $cf(\zeta) < \kappa$, then $\mid C_\zeta \mid < \kappa$;

(3) if γ is a limit point of C_ζ, then $C_\gamma = C_\zeta \cap \gamma$.

(a) Prove that if $cf(\zeta) = \kappa$, then the order type of C_ζ is κ.

(b) Suppose λ is a cardinal. A subset $X \subseteq \lambda$ is called *non-reflecting* in λ if the derived set $\{\delta \in \lambda : cf(\delta) > \aleph_0 \text{ and } X \cap \delta \text{ is stationary in } \delta\}$ is empty.
Prove that the set $E_\xi = \{\zeta : C_\zeta \text{ has order type } \xi\}$ is non-reflecting in κ^+ for every $\xi \leqslant \kappa$. Deduce that $\Box_\kappa$ implies $lim(\kappa^+)$ is the disjoint union of κ non-reflecting subsets.

6

(i) Let G be a generic filter in a forcing $\mathbb{P}$ over a countable transitive model $\mathbb{M}$. Define or state carefully any two of the following three italicized concepts or results:

(a) the *Forcing Theorem* for $\mathbb{M}$ and $\mathbb{M}[G]$;

(b) the *canonical name* $\dot{a}$ for an element $a \in \mathbb{M}$;

(c) the *Generalized* Δ*-System Lemma.*

(ii) Suppose that G is a generic filter in the forcing $\mathbb{P}$ over the countable transitive model $\mathbb{M}$, and $\mathbb{P}$ has the κ-chain condition in $\mathbb{M}$, where $\kappa = cf(\kappa) > \aleph_0$.

(a) Let $\alpha \in Ord$, $\beta \in Ord$, $\alpha \geqslant \omega$. Suppose $\mathbb{M}[G] \models (f : \alpha \to \beta$ is a function$)$. Prove that there is a function $y \in \mathbb{M}$, $dom(y) = \alpha$, such that $((\forall \zeta < \alpha)(\mid y(\zeta) \mid < \kappa))^{\mathbb{M}}$, and $\mathbb{M}[G] \models (\forall \zeta < \alpha)(f(\zeta) \in y(\zeta))$.

(b) Deduce that forcing with $\mathbb{P}$ preserves cardinalities $\geqslant \kappa$.

(iii) (a) Suppose that $\mathbb{M}$ is a countable transitive model, $(\mathbb{P}$ is a c.c.c. forcing$)^{\mathbb{M}}$, and $\mathbb{M}[G] \models \diamondsuit$. Stating any combinatorial equivalences to which you appeal, if any, prove $\mathbb{M} \models \diamondsuit$.

(b) Suppose $\lambda = cf(\lambda) > \aleph_0$. Show that it is relatively consistent that $\diamondsuit_\lambda$ fails.

(iv) Suppose $S \subseteq Ord$. Define $Levy(\aleph_0, S) = \{p : p$ is a function, $\mid p \mid < \aleph_0$, $dom(p) \subseteq \aleph_0 \times S$, $(\forall (n, \alpha) \in dom(p))(p(n, \alpha) = 0$ or $p(n, \alpha) < \alpha)\}$, and $p \leqslant_{Levy(\aleph_0, S)} q$ if and only if $p \subseteq q$.
Suppose λ is a strongly inaccessible cardinal. Stating fully any pertinent ancillary propositions, prove that $Levy(\aleph_0, \lambda)$ has the λ-chain condition.
Let G be a generic filter in $Levy(\aleph_0, \lambda)$ over a countable transitive model $\mathbb{M}$. Show that $\aleph_1^{\mathbb{M}[G]} = \lambda$.

END OF PAPER