UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 9:00 am to 12:00 pm

PAPER 18

CATEGORY THEORY

*Attempt no more than **FIVE** questions.*

*There are **EIGHT** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{C}$ be a locally small category.

(a) Define the *Yoneda embedding* $Y\colon \mathcal{C}^{\mathrm{op}} \to [\mathcal{C}, \mathrm{Set}]$. State and prove the Yoneda Lemma. State the properties of the Yoneda embedding which follow from the Yoneda Lemma.

(b) Define a *representation* (A, x) of a functor $F\colon \mathcal{C} \to \mathrm{Set}$. Prove that representations are unique up to unique isomorphism.

2

(a) Define the terms *monomorphism*, *epimorphism*, *strong monomorphism* and *regular monomorphism*. Show that a regular monomorphism is indeed a monomorphism.

A morphism $f\colon A \to B$ is called a **strict monomorphism** if every morphism $g\colon C \to B$ satisfying $(hf = kf \Rightarrow hg = kg)$ factors uniquely through f. Prove that every regular monomorphism is strict and that every strict monomorphism is strong.

(b) Define a *split coequaliser diagram* and prove that it is indeed a coequaliser. Recall that $e\colon E \to E$ is **idempotent** if $ee = e$, and that an idempotent **splits** if it can be factored as $e = fg$ where gf is an identity morphism. Prove that an idempotent e splits if and only if the pair $(e, 1_E)$ has a coequaliser.

3

Throughout this question, let $\mathcal{C}$ be a locally small and complete category.

(a) Define what it means for a functor $F\colon \mathcal{C} \to \mathcal{D}$ to *preserve*, *reflect* or *create* limits of shape $\mathcal{J}$. Prove that representable functors $\mathcal{C}(A, -)\colon \mathcal{C} \to \mathrm{Set}$ preserve all small limits.

(b) Now let $\mathcal{D}$ be any category. Prove that the functor category $[\mathcal{D}, \mathcal{C}]$ has all small limits and that the forgetful functor $U\colon [\mathcal{D}, \mathcal{C}] \to \mathcal{C}^{\mathrm{ob}\mathcal{D}}$ creates them.

[You are expected to show that what you claim to be a limit in $[\mathcal{D}, \mathcal{C}]$ really is a limit.]

4

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

(a) Define what it means for two functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ to be *adjoint*. Given an adjunction $F \dashv G$, define the *unit* η and the *counit* ϵ of the adjunction. State how the unit and counit determine the correspondence of the adjunction. Prove that η is a natural transformation. State and prove the triangular identities for η and ϵ.

(b) Prove that, given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ and natural transformations $\eta\colon 1_{\mathcal{C}} \to GF$ and $\epsilon\colon FG \to 1_{\mathcal{D}}$ satisfying the triangular identities, there is a unique adjunction between F and G with unit η and counit ϵ.

(c) Let $\mathcal{C}$ be a small category. Show that the functor $- \times \mathcal{C}\colon \mathrm{Cat} \to \mathrm{Cat}$ is left adjoint to $[\mathcal{C}, -]\colon \mathrm{Cat} \to \mathrm{Cat}$. What are the unit and counit of this adjunction?

[In this part of the question, you are not expected to check details such as well-definedness or the naturality of the bijection.]

5

Consider a functor $G\colon \mathcal{C} \to \mathrm{Set}$ with $\mathcal{C}$ locally small and complete.

(a) Define, for each object $X \in \mathrm{Set}$, the category $(X \downarrow G)$.

(b) Show that G is representable if and only if $(1 \downarrow G)$ has an initial object.

(c) Show that a complete, locally small category has an initial object if and only if it has a weakly initial set.

(d) Deduce that $G\colon \mathcal{C} \to \mathrm{Set}$ is representable if and only if it preserves limits and $(1 \downarrow G)$ has a weakly initial set.

[You may assume results about limits in $(X \downarrow G)$, provided they are stated clearly. You may also assume that representable functors preserve limits.]

6

Let $\mathbb{T} = (T\colon \mathcal{C} \to \mathcal{C}, \eta\colon 1_{\mathcal{C}} \to T, \mu\colon TT \to T)$ be a monad on the category $\mathcal{C}$.

(a) Define the category of algebras $\mathcal{C}^{\mathbb{T}}$ and a free $\mathbb{T}$-algebra on an object A of $\mathcal{C}$.

Prove carefully that the Eilenberg–Moore adjunction $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ is terminal in the category of adjunctions inducing the monad $\mathbb{T}$.

(b) Let $\mathcal{C} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$ be an adjunction inducing $\mathbb{T}$. Prove that the comparison functor $K\colon \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is full and faithful when the standard free presentation

$$FGFGB \underset{\epsilon_{FGB}}{\overset{FG\epsilon_B}{\rightrightarrows}} FGB \xrightarrow{\epsilon_B} B$$

is a coequaliser for all objects B in $\mathcal{D}$.

[You may assume any standard results from the course, provided they are clearly stated.]

7

State and prove the Crude Monadicity Theorem.

[You may use any standard results from the course, provided they are clearly stated.]

8

Let $\mathcal{A}$ be an abelian category.

(a) Given a commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccc} K & \xrightarrow{f} & A & \xrightarrow{g} & B \\ {\scriptstyle k}\downarrow & (1) & \downarrow{\scriptstyle a} & (2) & \downarrow{\scriptstyle b} \\ K' & \xrightarrow[f']{} & A' & \xrightarrow[g']{} & B' \end{array}$$

with $f = \ker g$ and $f' = \ker g'$, prove that if b is a monomorphism then the square (1) is a pullback. Prove also that if the square (2) is a pullback, then k is an isomorphism.

(b) Define the image factorisation of a morphism $f\colon A \to B$ in $\mathcal{A}$ and prove that it is unique (up to isomorphism) and functorial (i.e. prove that image factorisation gives rise to a functor $I\colon \mathrm{Arr}\mathcal{A} \to \mathcal{A}$ from the arrow-category on $\mathcal{A}$ to $\mathcal{A}$). Prove also that image factorisation is stable under pullback: Given a diagram

$$\begin{array}{ccccc} A' & \xrightarrow{p'} & I' & \xrightarrow{i'} & B' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[p]{} & I & \xrightarrow[i]{} & B \end{array}$$

where both squares are pullbacks, prove that if the bottom row is the image factorisation of its composite $f = ip$ then the top row is the image factorisation of the pullback of f along b.

[You may use any standard results about pullbacks, provided they are stated clearly.]

END OF PAPER