UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 6 June, 2013 9:00 am to 12:00 pm

PAPER 17

COMPLEX MANIFOLDS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Given a codimension one closed complex submanifold V of a complex manifold M, describe the corresponding holomorphic line bundle $[V]$, and justify the fact that it is well-defined up to isomorphism, i.e. independent of any choices you may have made. [Throughout this question, the standard bijection between (isomorphism classes of) holomorphic line bundles on M and invertible $\mathcal{O}_M$-modules may be assumed without proof.]

Define the *canonical line bundle* K_M on a complex manifold M; when $M = \mathbf{P}^n(\mathbf{C})$, find an expression for K_M in terms of the hyperplane bundle $[H]$.

Let $\phi : X \to Y$ be a holomorphic map of complex manifolds and $\pi : L \to Y$ be a holomorphic line bundle on Y. For $U \subset X$ open, we define $\mathcal{F}(U)$ to be the set of holomorphic maps $\theta : U \to L$ such that $\pi \circ \theta = \phi|_U$; show that this determines an invertible $\mathcal{O}_X$-module $\mathcal{F}$. We denote the corresponding holomorphic line bundle on X by $\phi^* L$.

Suppose $x_1, \ldots, x_n$ denote the standard coordinates on $\mathbf{C}^n$, and consider the projective space $\mathbf{P}^{n-1}(\mathbf{C})$, with homogeneous coordinates $Y_1, \ldots, Y_n$. Let $X \subset \mathbf{C}^n \times \mathbf{P}^{n-1}(\mathbf{C})$ be the subset defined by the equations $x_i Y_j = x_j Y_i$ for all $1 \leqslant i, j \leqslant n$. Show that X is a complex manifold. If π_1 denotes the projection map of X to $\mathbf{C}^n$, show that $\pi_1^{-1}(\mathbf{0})$ is a submanifold E of X biholomorphic to $\mathbf{P}^{n-1}(\mathbf{C})$, and that the restriction of π_1 to $X \setminus E$ is a biholomorphism to $\mathbf{C}^n \setminus \{\mathbf{0}\}$. Prove that $K_X \cong [E]^{\otimes(n-1)}$.

For any complex manifold M of dimension n and point $P \in M$, sketch briefly how, using the above construction, you would construct a complex manifold $\tilde{M}$ and a holomorphic map $\phi : \tilde{M} \to M$, with $E := \phi^{-1}(P)$ a submanifold biholomorphic to $\mathbf{P}^{n-1}(\mathbf{C})$ and $\phi : \tilde{M} \setminus \phi^{-1}(P) \to M \setminus \{P\}$ a biholomorphic map (you need not show that your construction is independent of the choices that you may have made). Prove that

$$K_{\tilde{M}} \cong \phi^* K_M \otimes [E]^{\otimes(n-1)}.$$

2

Define what is meant by a connection D on a complex vector bundle E over a smooth manifold M, and show that connections on E always exist. What is the *curvature* Θ_D associated to the connection? State and prove Cartan's equation for the corresponding curvature matrix Θ (with respect to a given local frame for E). Explain how one obtains a globally defined closed 2-form $\mathrm{Tr}\ \Theta_D$ from the curvature.

If E has rank r and D is a connection on E, show that there is a naturally defined associated connection $D^{(r)}$ on $\bigwedge^r E$, whose curvature 2-form is just $\mathrm{Tr}\ \Theta_D$. Justify briefly (omitting details) why the De Rham cohomology class determined by $\mathrm{Tr}\ \Theta_D$ is independent of the connection.

Suppose now that M is a complex manifold and that E is a holomorphic vector bundle over M, equipped with a hermitian metric. If $h = (h_{ij})$ is the hermitian matrix defined by $h_{ij} := (e_i, e_j)$ for some local holomorphic frame $e_1, \ldots, e_r$ for E, we define a matrix of 2-forms locally by

$$(\bar{\partial}\partial h).h^{-1} + (\partial h).h^{-1} \wedge (\bar{\partial} h).h^{-1},$$

where dot here denotes matrix multiplication and wedge denotes the product of matrices of 1-forms. Show that the trace of the above matrix is an exact local 2-form independent of the choice of holomorphic frame, and hence defines a global closed 2-form α. Why is the De Rham cohomology class of α independent of the choice of metric? Justify your answer.

3

Explain what is meant by a *Hodge metric* on a compact complex manifold. Show that a complex torus admits a Hodge metric if and only if it admits an *invariant* one — here, you should explain what is meant by the word *invariant* in this context.

Given a lattice Λ in $\mathbf{C}^n$ and a choice of basis $\lambda_1, \ldots, \lambda_{2n}$ for Λ, we can take the standard basis $e_1, \ldots, e_n$ for $\mathbf{C}^n$ and define the *period matrix* $\Omega = (\omega_{\alpha i})$ to be the $n \times 2n$ matrix such that $\lambda_i = \sum_{\alpha=1}^{n} \omega_{\alpha i}\, e_\alpha$. Prove that the complex torus $M = \mathbf{C}^n/\Lambda$ admits a Hodge metric if and only if there exists an integral skew-symmetric matrix Q satisfying $\Omega\, Q^{-1}\, \Omega^t = 0$ and $-i\Omega\, Q^{-1}\, \overline{\Omega}^t$ is a positive-definite (hermitian) matrix.

Suppose now $\Lambda \subset \mathbf{C}^2$ is a lattice with period matrix $\Omega = (\mathrm{I}_2, iA)$, where I_2 denotes the 2×2 identity matrix and $A = \begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$. Prove that $M = \mathbf{C}^2/\Lambda$ does not admit any Hodge metric.

4

For M a complex manifold of dimension n equipped with a hermitian metric, let ω denote the associated $(1,1)$-form. Give the defining property of the Hodge $*$-operator

$$* : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{n-p,n-q}.$$

For $\psi \in \mathcal{A}_M^{p,q}(U)$, state a formula for $** \psi$. Let $L : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p+1,q+1}$ denote the operator given by $L(\eta) = \omega \wedge \eta$; define the formal adjoint $\Lambda = L^* : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p-1,q-1}$. When M is compact, show that L and Λ are adjoint operators when acting on global forms (with induced inner-product).

Assuming the result that the commutator $[L, \Lambda]$ acts via multiplication by $k - n$ on $\mathcal{A}_M^k$, prove by induction that for $r \geqslant 1$, $[L^r, \Lambda] = \{r(k-n) + r(r-1)\}L^{r-1}$ as operators on $\mathcal{A}_M^k$. Deduce by induction on $r + k$ that if r, k are non-negative integers with $r + k \leqslant n$, then $L^r : \mathcal{A}_M^k \to \mathcal{A}_M^{k+2r}$ is an injective morphism of sheaves. [*Hint: Prove the result first for* $k = 0, 1$. *For* $k > 1$, *under an appropriate inductive hypothesis show that, for any* $\psi \in \mathcal{A}_m^k(U)$ *with* $L^r\psi = 0$, *we can write* $\psi = L\alpha$ *for some* $\alpha \in \mathcal{A}_m^{k-2}(U)$ *with* $L^{r+1}\alpha = 0$.]

In the case when M is a Kähler manifold, state the Hodge identities for $[\Lambda, \bar{\partial}]$ and $[\Lambda, \partial]$; deduce corresponding identities for $[L, \bar{\partial}^*]$ and $[L, \partial^*]$. Deduce also identities between the various Laplacians Δ_d, Δ_∂ and $\Delta_{\bar{\partial}}$ on M. In this case, show that the Laplacians commute with L.

For M is a compact Kähler manifold, explain briefly why the Hodge decomposition theorem for global forms implies that the Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(M)$ is isomorphic to the space $\mathcal{H}^{p,q}(M)$ of harmonic (p,q)-forms. If $h^{p,q}$ denotes the complex dimension of $H_{\bar{\partial}}^{p,q}(M)$, prove that $h^{p,q} = h^{q,p}$ and $h^{p,q} = h^{n-p,n-q}$. Deduce that $L^{n-(p+q)}$ induces an isomorphism between Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}(M) \to H_{\bar{\partial}}^{n-q,n-p}(M)$.

END OF PAPER