UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 31 May, 2013 9:00 am to 12:00 pm

PAPER 16

SPECTRAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what it means for a map $\pi : M \to N$ between Riemannian manifolds to be a Riemannian submersion with totally geodesic fibres. For such a map, state and prove the relation between the actions of the Laplacians of M and N on relevant functions.

Derive, with proof, an expression for the eigenspaces for the product $(M \times N, k)$ of two Riemannian manifolds (M, g) and (N, h), with the product metric $k = g \times h$, in terms of the eigenspaces of M and N.

You may quote without proof, but should state clearly, any subsidiary results that you require from Analysis or Riemannian Geometry.

2

Show how the subspace $T \leqslant {\mathbb{F}_3}^4$ spanned by the vectors $(0, 1, 1, 1)$ and $(1, 1, -1, 0)$ may be used to construct two lattices L^+ and L^- in $\mathbb{R}^4$ such that the flat tori $\mathbb{R}^4/L^{\pm}$ have the same length spectrum but are not isometric.

3

Define a heat kernel for a Riemannian manifold (M, g) and a parametrix for the heat operator.

Stating clearly, but without proof, any relevant extendability and differentiability properties that are required, prove that a parametrix determines a (global) heat kernel on M.

Obtain, with proof, an explicit expression for the heat kernel in terms of eigenvalues and eigenfunctions of the Laplacian.

4

Define a nowhere homogeneous or 'bumpy' metric on a differential manifold M^m and state Sunada's Lemma.

State the dimension of the k-jet space for maps from M^m to another differential manifold N^n, and identify its fibre as a bundle over $M \times N$.

Given open sets U_i and U_j of M^m such that each of $\overline{U}_i$ and $\overline{U}_j$ is diffeomorphic with a closed ball in $\mathbb{R}^m$, let $\mathcal{S}_{ij}$ be the set of metrics g on M such that $(\overline{U}_i, g)$ is isometric with $(\overline{U}_j, g)$. Prove that the complement $\mathcal{CS}_{ij}$ of $\mathcal{S}_{ij}$ is dense in the space of all metrics on M with the $\mathcal{C}^\infty$-topology.

5

Define a partition of a closed Riemann surface S and state Bers' Theorem.

Derive parameters that determine S up to isometry and define Teichmüller space.

State Wolpert's Theorem and Buser's Theorem for simplifying its proof.

Stating without proof, but defining any terms involved, any subsidiary results that you require, prove Buser's Theorem.

END OF PAPER