UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 9:00 am to 12:00 pm

PAPER 15

DIFFERENTIAL GEOMETRY

*You may attempt **ALL** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the definition of *smooth manifold*.

Give an example (with justifications!) of a connected, second countable, para-compact, Hausdorff topological space X that does *not* admit the structure of a smooth manifold.

Given two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, show that their topological product $\mathcal{M} \times \mathcal{N}$ admits a natural smooth manifold structure.

Let $\mathcal{M}$ be a manifold and $x \in \mathcal{M}$. State the definition of *tangent space* $T_x\mathcal{M}$ at x, and show that $T_x\mathcal{M}$ is a vector space of dimension the dimension of $\mathcal{M}$. Show that a smooth map $f : \mathcal{M} \to \mathcal{N}$ of manifolds gives rise to a map $(f_*)_x : T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$ known as the *push forward* (or differential map).

With the notation of above, suppose now in addition that $\mathcal{M}$ is connected and $(f_*)_x = 0$ *for all* x. Show that f is a constant map.

2

State the definition of *Riemannian metric*. Show that a Riemannian metric on a smooth manifold $\mathcal{M}$ defines a bundle isomorphism between the tangent and cotangent bundles.

Show that given any Riemannian manifold $(\mathcal{M}, g)$, a point $p \in \mathcal{M}$, and a non-empty open neighbourhood $\mathcal{U}$ with $p \in \mathcal{U}$, then there exists a second Riemannian metric $\widetilde{g}$ on $\mathcal{M}$ such that $\widetilde{g} = g$ in $\mathcal{M} \setminus \mathcal{U}$ and such that $(\mathcal{M}, \widetilde{g})$ is locally isometric to Euclidean space in some neighbourhood of p.

State what it means for a Riemannian manifold to be *geodesically complete*.

Let $\mathcal{M}$ be a manifold such that $\mathcal{M} \setminus X$ is diffeomorphic to $\mathbb{R}^n \setminus B$ where B is the closed ball of radius 1, for some *compact* subset $X \subset \mathcal{M}$. Let g be a Riemannian metric on $\mathcal{M}$, such that with respect to the local co-ordinates x^i induced on $\mathcal{M} \setminus X$ from the standard coordinates of $\mathbb{R}^n \setminus B$, $g_{ij} = e_{ij}$ outside some sufficiently large closed ball, where $e_{ij} = 1$ if $i = j$ and 0 otherwise. Show that $(\mathcal{M}, g)$ is geodesically complete.

Now let $\mathcal{M}$ be a smooth manifold, and g and $\widetilde{g}$ be two metrics on $\mathcal{M}$ such that $\widetilde{g}(v, v) \geqslant g(v, v)$ for all vectors $v \in T\mathcal{M}$. Show that if g is geodesically complete, so is $\widetilde{g}$.

3

Let $(\mathcal{M}, g)$ be a Riemannian manifold. Show that there exists a unique connection ∇ in $T\mathcal{M}$ (the so-called *Levi-Civita* connection) such that

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$\nabla_X Y - \nabla_Y X = [X,Y].$$

Give an expression for ∇ in local coordinates in terms of g.

Let $\mathcal{M}$ be connected and g, $\widetilde{g}$ denote two Riemannian metrics on $\mathcal{M}$ such that

$$\nabla = \widetilde{\nabla}$$

as maps:

$$T\mathcal{M} \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M}),$$

where ∇, $\widetilde{\nabla}$ denote the Levi-Civita connection of g, $\tilde{g}$ respectively. Suppose, moreover, that $g|_x = \widetilde{g}|_x$ at some $x \in \mathcal{M}$. Show that $g = \widetilde{g}$.

What if one drops the assumption $g|_x = \widetilde{g}|_x$?

4

Let $(\mathcal{M}, g)$ be a Riemannian manifold. Define the *Riemann curvature tensor* and show that it indeed defines a tensor.

Define *sectional* and *Ricci curvature*.

Compute the Riemann curvature tensor of the standard unit n-sphere $\mathbb{S}^n$ with its induced metric from Euclidean $\mathbb{R}^{n+1}$. Show that this metric on $\mathbb{S}^n$ is *Einstein*, i.e.

$$\mathrm{Ric} = \Lambda g$$

for some constant Λ which you should compute.

5

State and prove the *Bonnet–Myers* theorem concerning complete Riemannian manifolds with Ricci curvature bounded below.

Show by explicit example that the assumption of geodesic completeness is necessary.

END OF PAPER