UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 4 June, 2013 1:30 pm to 4:30 pm

PAPER 14

ALGEBRAIC TOPOLOGY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Compute the integral cohomology rings of the following spaces.

1. $\mathbb{CP}^8/\mathbb{CP}^3$.
2. $(S^2 \times S^2) \cup_f D^3$ where $f : S^2 \to S^2 \times S^2$ is given by $f(x) = (x, x)$.
3. $(S^2 \times S^2)/\sim$, where $(\mathbf{x}, \mathbf{y}) \sim (-\mathbf{x}, -\mathbf{y})$.

2

Suppose that (C_*, d_C) and (D_*, d_D) are finitely generated free chain complexes defined over $\mathbb{Z}$, and that $f : C_* \to D_*$ is a chain map. Let $M_i = C_{i-1} \oplus D_i$, and define $d_f : M_i \to M_{i-1}$ by

$$d_f(x, y) = (d_C x, (-1)^i f(x) + d_D y).$$

Show that (M, d_f) is a chain complex, and that $H_*(M) = 0$ if and only if the map $f_* : H_*(C) \to H_*(D)$ is an isomorphism.

If $f : X \to Y$ is a map of finite cell complexes, show that if $f_* : H_*(X; \mathbb{Z}/p) \to H_*(Y; \mathbb{Z}/p)$ is an isomorphism for each prime p, then $f_* : H_*(X; \mathbb{Z}) \to H_*(Y; \mathbb{Z})$ is an isomorphism.

3

Show that any map $f : \mathbb{CP}^2 \to S^2 \times S^2$ has degree 0.

If $g : S^2 \times S^2 \to \mathbb{CP}^2$ has degree n, what are the possible values of n? Construct a map of each possible degree.

If $h : S^2 \times S^2 \to \mathbb{CP}^2 \# \mathbb{CP}^2$ has degree n, what are the possible values of n? Construct a map of each possible degree.

4

State the Thom isomorphism theorem for unoriented real vector bundles and derive the (unoriented) Gysin sequence from it. Compute the ring structure on $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$. (You may assume the groups $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ are known.)

Now assume that $E \xrightarrow{\pi} B$ is an n-dimensional oriented real vector bundle. Define the Euler class of E. If $U \in H^n(D(E), S(E))$ is the Thom class, show that $U \cup U = U \cup \pi^*(e(E))$. Deduce that if n is odd, $2e(E) = 0$.

5

Let G be a topological group (*i.e.* the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are continuous) with identity element e. If $\alpha, \beta : (I^n, \partial I^n) \to (G, e)$, define $\alpha\beta : (I^n, \partial I^n) \to (G, e)$ by $\alpha\beta(x) = \alpha(x)\beta(x)$. Show that $[\alpha\beta] = [\alpha] + [\beta]$ in $\pi_n(G, e)$.

Identifying S^3 with the unit quaternions, define $\Phi_{k,l} : S^3 \times S^3$ by $\Phi_{k,l}(q_1, q_2) = (q_1^k q_2 q_1^l, q_1)$. Show that Φ is a homeomorphism, and compute the induced map $\Phi_* : H_*(S^3 \times S^3) \to H_*(S^3 \times S^3)$.

If $X_{k,l} = S^3 \times D^4 \cup_{\Phi_{k,l}} D^4 \times S^3$, compute $H_*(X_{k,l})$.

END OF PAPER