UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 30 May, 2013 1:30 pm to 4:30 pm

PAPER 13

ALGEBRAIC GEOMETRY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let A and B be commutative rings, $X = \text{Spec } A$ and $Y = \text{Spec } B$. Give the proof of the theorem which states that there is a 1-1 correspondence between the set of ring homomorphisms $A \to B$ and morphisms $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ as locally ringed spaces. Describe Spec $\mathbb{R}[t_1, t_2]$ and the morphism Spec $\mathbb{C}[t_1, t_2] \to$ Spec $\mathbb{R}[t_1, t_2]$ induced by the homomorphism $\mathbb{R} \to \mathbb{C}$ (with justifications).

2

Give the definitions of an irreducible scheme, a reduced scheme, and an integral scheme. Let A be a commutative ring. Show that A is an integral domain if and only if Spec A is reduced and irreducible. Let X be a scheme. Show that X is integral if and only if it is reduced and irreducible. Next, let X be an integral scheme and U a non-empty open subscheme. Give the definitions of the generic point and the function field K of X, and show that the natural map $\mathcal{O}_X(U) \to K$ is injective. Now, find a morphism $f : X \to Y$ of integral schemes and a point $y \in Y$ such that the fibre of f over y is neither reduced nor irreducible.

3

Let $f : X \to Y$ be a morphism of Noetherian schemes and $\mathcal{F}$ a quasi-coherent sheaf on X. Give the proof of the theorem which states that $f_*\mathcal{F}$ is quasi-coherent. Find an example in which $\mathcal{F}$ *is not* coherent but $f_*\mathcal{F}$ *is* coherent. Show that there is no such example if f is a finite morphism. Next find an example in which f is an open immersion of integral schemes but not an isomorphism and that $f_*\mathcal{O}_X$ is coherent.

[Give justifications for your examples.]

4

Let X be a Noetherian scheme such that the intersection of any two open affine subschemes is again affine. Let U be an open affine subscheme, $f : U \to X$ the inclusion morphism, and let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of quasi-coherent sheaves on U. Show that

$$0 \to f_*\mathcal{M}' \to f_*\mathcal{M} \to f_*\mathcal{M}'' \to 0$$

is also an exact sequence. Now let $\mathcal{U} = (U_i)_{i \in I}$ be a finite open affine cover of X, and $\mathcal{F}$ a quasi-coherent sheaf on X. Give the proof of the theorem which states that $\check{H}^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$ for every p.

5

Let $X = \mathbb{P}^3_{\mathbb{C}} = \text{Proj } \mathbb{C}[t_0, \ldots, t_3]$ and $W = \text{Spec } \mathbb{C}$. Calculate the cohomology groups and the Euler characteristic of the sheaf $\Omega_{X/W}$.

Now let F and G be irreducible homogeneous polynomials in the t_i with $\deg F = 5$ and $\deg G = 7$. Let Z be the closed subscheme of X defined by the ideal $\langle F, G \rangle$. Show that $\dim_{\mathbb{C}} H^1(Z, \mathcal{O}_Z) \geqslant 141$.

END OF PAPER