UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2013 1:30 pm to 4:30 pm

PAPER 12

EXTREMAL GRAPH THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State the Erdős–Stone theorem, giving a sufficient condition under which a graph G of order n will contain $K_{r+1}(t)$, where $t \geqslant \lfloor d \log n \rfloor$ and d does not depend on n. Prove the theorem.

State and prove a corresponding upper bound for t, with the same dependence on n.

Let H_k be the graph of order k obtained from a cycle of length k by adding all edges between vertices distance 2 apart on the cycle. For which values of k does the following hold: for every sequence $(G_n)_{n=1}^{\infty}$ of graphs with $|G_n| = n$ and $e(G_n) \geqslant 0.51\binom{n}{2}$, all but finitely many G_n contain H_k?

2

Define the parameter $\pi(H)$ of an ℓ-uniform hypergraph H. Show that there is a function $\delta = \delta(H, \epsilon)$ such that every ℓ-uniform hypergraph G of order n with $e(G) \geqslant (\pi(H) + \epsilon)\binom{n}{\ell}$ contains at least $\lfloor \delta n^{|H|} \rfloor$ copies of H.

Let G be a graph of order n and let $k_p(G)$ be the number of copies of K_p in G. Let $c \in \mathbb{R}$ and let $f(G) = k_2(G) - ck_3(G)$. Show that, amongst graphs of order n, the function $f(G)$ takes its maximum on some complete multipartite graph.

By writing $f(G) \leqslant \sum\limits_{i<k} a_{ij} - c \sum\limits_{i<j<k} a_i a_j a_k$, where $a_1, \ldots, a_q$ are integers and $\sum_{i=1}^{q} a_i = n$, show that $f(G)$ takes its maximum on a Turán graph.

By allowing the a_i to take rational values, deduce that $f(G) \leqslant \binom{q}{2}(n/q)^2 - c\binom{q}{3}(n/q)^3$ holds for some integer $q \geqslant 2$. Hence show that $k_3(G) \geqslant (4n/9)(k_2(G) - n^2/4)$.

3

Let G be a graph of order n. Show that, if G is K_{r+1}-free and $e(G) \geqslant t_r(n) - k$, then there is a complete r-partite graph H, with $V(H) = V(G)$ and $|E(H) \triangle E(G)| \leqslant 3k$.

Suppose that G contains no cycle of length 2013. Prove that, for each $\epsilon > 0$, there is a number $n_0 = n_0(\epsilon)$ so that, if $n > n_0$, the following statement holds: if G contains no cycle of length 2013 and $e(G) \geqslant (1/2 - \epsilon)\binom{n}{2}$, then there is a complete bipartite graph H, with $V(H) = V(G)$ and $|E(H) \triangle E(G)| \leqslant 11\epsilon\binom{n}{2}$.

[*Szemerédi's Regularity Lemma and its standard consequences may be used without proof, provided they are clearly stated.*]

Explain why the statement remains true if the condition "G contains no cycle of length 2013" is weakened to "G contains at most δn^{2013} cycles of length 2013, for some small constant δ depending on ϵ".

4

Let $\mathcal{Q}$ be a monotone graph property and let $\mathcal{Q}_n$ consist of those graphs in $\mathcal{Q}$ having vertex set $[n]$. Prove that $|\mathcal{Q}_n| = 2^{(1-1/r+o(1))\binom{n}{2}}$, where $r+1 = \min\{\chi(F) : F \notin \mathcal{Q}\}$.

[*Szemerédi's Regularity Lemma and its standard consequences may be used without proof, provided they are clearly stated.*]

Define the properties $\mathcal{C}(a,b)$ and define the parameter $r(\mathcal{P})$ of a hereditary graph property $\mathcal{P}$. What is $r(\mathcal{C}(a,b))$?

Briefly describe the main changes that must be made to your earlier proof in order to prove that $|\mathcal{P}_n| = 2^{(1-1/r+o(1))\binom{n}{2}}$, where $r = r(\mathcal{P})$.

Show that, if $\mathcal{P}$ and $\mathcal{Q}$ are both monotone, then $r(\mathcal{P} \cap \mathcal{Q}) = \min\{r(\mathcal{P}), r(\mathcal{Q})\}$. By considering $\mathcal{C}(2,0)$ and $\mathcal{C}(2,2)$, or otherwise, show that this equation does not hold for hereditary properties in general.

5

Let $\prod_{i=1}^n A_i$ be a product of finite probability spaces. Let $f : \prod_{i=1}^n A_i \to \mathbb{R}$ be such that $|f(z) - f(z')| \leqslant c_i$ whenever z and z' differ only in the ith co-ordinate. Prove that $\Pr\{|f - \mathbb{E}f\| \geqslant t\} \leqslant 2\exp(-2t^2/\sum_{i=1}^n c_i^2)$ for each real $t \geqslant 0$.

Let $G \in \mathcal{G}(n,p)$ be a random graph, where p is constant. Let $\chi(G)$ be its chromatic number. Show that $\Pr\{|\chi(G) - \mathbb{E}\chi(G)| \geqslant \lambda\sqrt{n}\} \leqslant 2e^{-2\lambda^2}$ for each real $\lambda \geqslant 0$.

Briefly outline why $\mathbb{E}\chi(G) = (1+o(1))n/2\log_{1/(1-p)} n$, highlighting the main points of your argument.

END OF PAPER