UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 3 June, 2013 9:00 am to 12:00 pm

PAPER 11

ADDITIVE COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define the additive energy $E(A, A)$ of a finite set A of integers. Show that if A has small doubling constant then it has large additive energy, giving a precise statement of this fact.

State a result in the converse direction and show how it follows from a result about sumsets along a graph, which you should state precisely.

Suppose that Γ is a bipartite graph on vertex set $V \cup W$, where $|V| = m$ and $|W| = n$, with δmn edges joining vertices in V to vertices in W. Let $\epsilon > 0$ be a further parameter. Show that there is a subset $V' \subseteq V$ with $|V'| \geqslant \delta m/2$ such that between at least a proportion $1 - 2\epsilon/\delta^2$ of the ordered pairs of points $(v_1, v_2) \in V' \times V'$ there are at least ϵn paths $v_1 \to w \to v_2$ of length two.

Discuss, *very briefly*, how this result is important in the proof of the result about sumsets along a graph which you stated earlier.

2

Let $(\phi(n))_{n \in \mathbf{N}}$ be a sequence in $\mathbf{R}/\mathbf{Z}$. What does it mean for this sequence to be equidistributed? Discuss, without proof, the link between equidistribution and exponential sums of the form $\mathbf{E}_{n \leqslant N} e(m\phi(n))$. Deduce a criterion, in terms of exponential sums like this, for ϕ to be equidistributed.

Suppose that $(\phi(n))_{n \in \mathbf{N}}$ is not equidistributed. Show that there is an $h \neq 0$ such that the sequence $(\Delta_h \phi(n))_{n \in \mathbf{N}}$ is not equidistributed, where $\Delta_h \phi(n) = \phi(n) - \phi(n + h)$.

Show that $(n^2\sqrt{2})_{n \in \mathbf{N}}$ is equidistributed in $\mathbf{R}/\mathbf{Z}$.

3

Define the Gowers $U^3(N)$-norm of a function $f : \{1, \ldots, N\} \to \mathbf{C}$.

What is the $U^3(N)$-norm of the function $f_0(x) = e(\alpha x^2)$?

Briefly discuss the connection between the Gowers U^3-norm and 4-term arithmetic progressions.

Show that if $|f(x)| \leqslant 1$ for all x and if $\|f\|_{U^3(N)} \geqslant \delta$ then there is a set $H \subseteq \{-N, \ldots, N\}$, $|H| \gg \delta^C N$, such that the following hold:

1. If $h \in H$ then there is some $\theta(h) \in \mathbf{R}/\mathbf{Z}$ such that
$$|\mathbb{E}_{x \in G} \partial_h f(x) e(-\theta(h)x)| \gg \delta^C;$$
2. There are $\gg \delta^C N^3$ quadruples $h_1, h_2, h_3, h_4 \in H$ such that
$$h_1 + h_2 = h_3 + h_4$$
and
$$\theta(h_1) + \theta(h_2) = \theta(h_3) + \theta(h_4).$$

Give an explicit function $\theta(h)$ which satisfies this in the case $f = f_0$. Give, without proof, an example of a function θ satisfying condition (2) for some $\eta > 0$, but which does not agree with any function of form $\theta(h) = \alpha h + \beta$ for more than $o(N)$ values of $h \in \{-N, \ldots, N\}$.

[You may assume any form of the inverse theorem for the U^2-norm you wish.]

4

Let N be a prime, and let $R \subseteq \mathbf{Z}/N\mathbf{Z}$ be a set of size d. Let $\epsilon > 0$. Define the *Bohr set* $B(R, \epsilon)$, and show that it contains a nonzero element if $\epsilon > N^{-1/d}$. Hence show that $B(R, \epsilon)$ contains an arithmetic progression of length at least $\epsilon N^{1/d}$.

Suppose that $A \subseteq \{1, \ldots, N\}$ is a set of size $0.01N$. Show there is an absolute constant $c > 0$ such that the set $2A - 2A$ contains an arithmetic progression of length at least N^c, for N sufficiently large.

By considering random sets, or otherwise, so that the same conclusion is not necessarily true of the set A itself.

[Any basic facts about the discrete Fourier transform, and about the distribution of prime numbers, should be stated but need not be proven.]

5

Let A be a set of integers. Suppose that $|A| = N$ and that $|A + A| \leqslant KN$. Show that $|\ell A| \leqslant K^{\ell} N$ for every integer $\ell \geqslant 2$.

State and prove the Ruzsa covering lemma. Show that for fixed K and $\epsilon > 0$ there there is some $\ell_0 = \ell_0(K, \epsilon)$ such that $|\ell A| \leqslant K^{\epsilon \ell} N$ for all $\ell \geqslant \ell_0(K, \epsilon)$.

END OF PAPER