UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 9:00 am to 11:00 am

PAPER 10

COMBINATORICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

State and prove the Local LYM inquality. State the LYM inequality, and give two proofs: one using Local LYM and one using maximal chains.

State Sperner's lemma on antichains, and explain why it follows from the LYM inequality.

A set system $\mathcal{A} \subset [n]^{(r)}$ is called *intersection-free* if no distinct $A, B, C \in \mathcal{A}$ satisfy $A \cap B \subset C$. Prove that an intersection-free family has size at most $1 + \binom{r}{\lfloor r/2 \rfloor}$.

[*Hint: Use Sperner's lemma.*]

2

State the Kruskal-Katona theorem.

Let $1 \leqslant r < n/2$. State the Erdős-Ko-Rado theorem concerning intersecting families in $[n]^{(r)}$. Give two proofs: one using the Kruskal-Katona theorem and one using intervals in cyclic orderings.

Give a proof of the LYM inequality using intervals in cyclic orderings.

3

State and prove the vertex-isoperimetric inequality in the discrete cube (Harper's theorem).

What does it mean for a sequence of graphs to form a *Lévy family*? Prove that the sequence of discrete cubes $(Q_n)_{n=1}^{\infty}$ forms a Lévy family.

[Estimates on binomial coefficients may be quoted without proof, provided that they are precisely stated.]

If $A \subset Q_n$ is a down-set and is extremal for Harper's theorem (in other words, the inequality in Harper's theorem is an equality), does it follow that A is isomorphic to an initial segment of the simplicial order on Q_n? Give a proof or counterexample as appropriate.

4

State and prove the Frankl-Wilson theorem (on modular intersections).

Let $A \subset \mathcal{P}([n])$ be a family of sets such that, for some positive integer k, we have $|x \cap y| = k$ for all distinct $x, y \in A$. Prove that $|A| \leqslant n$.

[*Hint: Consider the characteristic vectors (indicator functions), over the reals, of the points of* A.]

END OF PAPER