UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 5 June, 2013 1:30 pm to 4:30 pm

PAPER 1

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let V be a finite dimensional complex vector space. Describe the Lie algebra structure on the space L of linear maps from V to V.

What does it mean for a Lie subalgebra L_1 of L to be (i) abelian, (ii) nilpotent, or (iii) soluble?

What is a flag of V?

Prove that if L_1 is soluble then there is a flag of V invariant under L_1. Deduce that L_1 has a nilpotent subalgebra L_2 with L_1/L_2 abelian.

2

Let V be a finite dimensional complex vector space. Define what it means for a linear map $\alpha : V \longrightarrow V$ to be (i) nilpotent or (ii) semisimple.

Given α show that there are unique nilpotent α_n and semisimple α_s such that $\alpha = \alpha_n + \alpha_s$ and $\alpha_n \alpha_s = \alpha_s \alpha_n$.

Let U and W be subspaces of the space of all linear maps $\alpha : V \longrightarrow V$, with $U \leqslant W$, and let M be the set of α such that $[\alpha, W] \leqslant U$. Show that if α in M satisfies $tr(\alpha\beta) = 0$ for all β in M then α is nilpotent.

3

Define what it means for a finite dimensional complex Lie algebra L to be semisimple.

Define the Killing form B_L and show that it is non-degenerate when L is semisimple.

What is a Cartan subalgebra H of a complex Lie algebra L? Show that H is a maximal abelian subalgebra when L is semisimple.

4

What is meant by a reduced root system of rank r with a base Δ. What is its Cartan matrix?

What are the reduced root systems of rank 2? In each case give a base and the Cartan matrix, and describe the Weyl group and Weyl chambers.

5

Let L be a semisimple complex Lie algebra with Cartan subalgebra H.

Define what is meant by a representation of L with primitive element of weight ω where ω lies in the dual space H^* of H.

Show that for each ω in H^* there is an irreducible representation with primitive element of weight ω.

Describe the finite dimensional irreducible representations when $L = sl_2$ giving the weight of a primitive element in each case.

6

What is meant by the complexification of a finite dimensional real Lie algebra L_0.

Show that L_0 is semisimple if and only if its complexification is semisimple.

Give examples of two non-isomorphic real Lie algebras with isomorphic complexifications.

What is a split semisimple Lie algebra? Are your two real Lie algebras split semisimple?

END OF PAPER