UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 9:00 am to 11:00 am

PAPER 9

PERCOLATION ON GRAPHS

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Let $k \geqslant 1$ and $\gamma_0 > 1/(k+1)$ be fixed, and let $\gamma = \gamma(n)$ satisfy

$$\gamma_0 \leqslant \gamma(n) \leqslant 1 - \frac{\omega(n)}{\log n},$$

where $\omega(n) \to \infty$. Show that for $p = \gamma(\log n)/n$ whp $G_{n,p}$ has a (huge) component with $n - (1+o(1))n^{1-\gamma}$ vertices, and every other component is a tree of order at most k.

(ii) Let $L = \{1, \ldots, \ell\}$, where $\ell = \lfloor n^{1/2} \rfloor$, and for a random graph process $\widetilde{G} = (G_{n,t})_{t=0}^{N}$ on $\{1, \ldots, n\}$, let τ be the minimal t such that in $G_{n,t}$ any two vertices of L are connected. Finally, let $\omega(n) \to \infty$. Show that whp

$$\big(\log n - \omega(n)\big)n/4 \leqslant \tau(\widetilde{G}) \leqslant \big(\log n + \omega(n)\big)n/4\,.$$

2

(i) Call a partition (V_1, V_2) of the vertex set V of a graph G with n vertices a *cut* if no edge of G joins V_1 to V_2. [Thus $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.] Show that if every component of G has at most $(k+1)n/k(2k+1)$ vertices then there is a cut (V_1, V_2) with

$$\max\{|V_1|, |V_2|\} \leqslant (k+1)n/(2k+1)\,.$$

(ii) Let $\{e_{i,j} : \ i = 1, 2, \ldots, \ j = 1, \ldots, k\}$ be an array of independent random variables, with each $e_{i,j}$ a uniformly distributed random edge of the complete graph with vertex set $[n] = \{1, \ldots, n\}$. For $\omega = (\omega_i)_1^m \in [k]^m = \{1, \ldots, k\}^m$, let G_ω^m be the graph with vertex set $[n]$ and edge set $\{e_{i,\omega_i} : \ 1 \leqslant i \leqslant m\}$. Show that there is an integer $c = c_k$ depending only on k such that

$$\lim_{n\to\infty} \mathbb{P}\big(L_1(G_\omega^{cn}) \geqslant 2n/3 \text{ for every } \omega \in [k]^{cn}\big) = 1\,,$$

where $L_1(G)$ is the maximal order of a component of G.

3

(i) Let $p(n) = \lambda/n = (1-\varepsilon)/n$, with $\varepsilon > 0$ constant. Show that whp

$$L_1(G_{n,p}) = (1 + o(1))\ell_0\,,$$

where $L_1(G)$ denotes the maximal order of a component of $G_{n,p}$, $\ell_0 = \lfloor (\log n)/\delta \rfloor$, and

$$\delta = -\log\left(\lambda e^{1-\lambda}\right) = -\log(1-\varepsilon) - \varepsilon = \varepsilon^2/2 + \varepsilon^3/3 + \cdots\,.$$

(ii) Deduce that for $\lambda > 1$ we have

$$\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}(\lambda e^{-\lambda})^k = \lambda^*\,,$$

where $0 < \lambda^* < 1$ is defined by $\lambda^* e^{-\lambda^*} = \lambda e^{-\lambda}$.

4

(i) Let $\mathcal{T}_{(n,p)}$ be the Galton–Watson branching process with offspring distribution $\mathrm{Bi}(n,p)$. Show that, for $p = (1+\varepsilon)/n$, with $\varepsilon > 0$ small, the survival probability $\rho = \rho_{(n,p)}$ of the binomial Galton–Watson branching process $\mathcal{T}_{n,p}$ satisfies

$$2\varepsilon - 4\varepsilon^2 \leqslant \rho \leqslant 2\varepsilon\,.$$

(ii) Let $p = (1+\varepsilon)/n$ with $n^{-1/6} \leqslant \varepsilon = \varepsilon(n) = o(1)$. Prove that $\mathbb{E}\big(L_1(G_{n,p})\big) = (2+o(1))\varepsilon n$, where $L_1(G)$ is the maximal order of a component of a graph G.

5

A rule R for an Achlioptas process $\widehat{G}_R = (G_t)$ is said to satisfy the *Explosive Percolation Hypothesis (EPH) with jump* $\delta > 0$ if whp we have

$$L_1(G_{t'_c}) = o(n) \quad \text{and} \quad L_1(G_{t_c}) \geqslant \delta n\,,$$

where $\delta n < t_c = t_c(n) < 3n$ and $t'_c = t_c(n) - \phi(n)$, with $\phi(n) = o(n)$.

(i) Prove that, given $\delta > 0$, there is a constant $D = D(\delta)$ such that if EPH holds with jump δ then, for every fixed $k \geqslant 1$, whp we have

$$N_{[k,Dk)}(G_{t_k}) \geqslant \delta n/4\,,$$

where $t_k = t_c - \delta n/2k$, and $N_{[k,\ell)}(G)$ denotes the total number of vertices on the components of G with at least k and fewer than ℓ vertices.

(ii) Without going into any detailed calculations, sketch how the first part can be used to show that no rule satisfies EPH.

END OF PAPER