UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 83

SET THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

You may assume the Axiom of Choice for all questions except question 1.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define xEy by $\mathcal{P}(x) \in \mathcal{P}(y)$. By considering a set with no E-minimal member or otherwise, prove without using the Axiom of Foundation that E is wellfounded.

(ii) If $\mathfrak{P} = \langle P, \leqslant \rangle$ is a poset, let $\mathfrak{P}^*$ be the poset of chains in $\mathfrak{P}$. By use of Hartogs' theorem or otherwise prove that there is no order-preserving injection $\mathfrak{P}^* \hookrightarrow \mathfrak{P}$.

2

HC, the set of hereditarily countable sets, is the $\subseteq$-least set X such that $\mathcal{P}_{\aleph_1}(X)$ (the class of countable subsets of X) is included in X. Explain why this definition makes sense.

By considering an injection $\mathcal{P}_{\aleph_1}(\Re) \hookrightarrow \Re$ or otherwise establish that $|HC| \leqslant 2^{\aleph_0}$.

What is the rank of HC? What axioms of ZFC does it model? (Brief expanations will suffice).

3

Show that every infinite set X has $2^{2^{|X|}}$ ultrafilters.

4

How many pairwise nonisomorphic total orders $\langle X, \leqslant_X \rangle$ can you find where $|X| = \aleph_1$ and every proper initial segment is isomorphic to the rationals (with possibly an endpoint)? [*Hint: stationary sets.*]

5

What are Aronszajn trees? Are there any?

END OF PAPER