UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 12 June, 2012 1:30 pm to 4:30 pm

PAPER 82

FOURIER ANALYSIS

*Attempt no more than **THREE** questions,*
*with at most **ONE** question from each section.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $f \in C(\mathbb{T})$ is a function, we define

$$S_N(f,t) := \sum_{j=-N}^{N} \hat{f}(j) e^{2\pi i j t}$$

and

$$\tilde{S}_N(f,t) := \sum_{j=-N}^{N} (1 - \frac{|j|}{N}) \hat{f}(j) e^{2\pi i j t}.$$

Write $S^*(f,t) = \sup_N |S_N(f,t)|$ and $\tilde{S}^*(f,t) = \sup_N |\tilde{S}_N(f,t)|$.

1. Show that if $\|f\|_\infty \leqslant 1$ then $\tilde{S}^*(f,t) \leqslant C$, where C is an absolute constant independent of f and t.
2. Show by contrast that for every open set $E \subseteq \mathbb{T}$ of measure δ there is a function $f \in C(\mathbb{T})$ with $\|f\|_\infty \leqslant 1$, but such that $S^*(f,t) \geqslant \psi(\delta)$ for all $t \in E$, where $\psi : \mathbb{R} \to \mathbb{R}$ is a function tending to ∞ as $\delta \to 0$.

2

Consider a sequence $(u_n)_{n=1}^N \subseteq \mathbb{R}/\mathbb{Z}$. Let δ, $0 < \delta < \frac{1}{10}$, be a small parameter. Define $f = \phi * \psi$, where $\phi = 1_{[\frac{1}{3}-\delta, \frac{1}{3}+\delta]}$ and $\psi = \frac{1}{2\delta} 1_{[-\delta,\delta]}$. Explain why S_N, the number of $n \leqslant N$ for which $\frac{1}{3} \leqslant u_n \leqslant \frac{2}{3}$, is at most $\sum_{n \leqslant N} f(u_n)$.

By considering the Fourier expansion of f, prove that for any choice of $M > 1$ we have

$$S_N - \frac{N}{3} \leqslant C\big(\delta N + \frac{N}{\delta M} + M \max_{0<|k|<M} |\sum_{n \leqslant N} e^{2\pi i k u_n}|\big),$$

where C is an absolute constant independent of N, f and δ.

Briefly justify why it is valid to expand f as a Fourier series (you may assume that, if all Fourier coefficients of a continuous function are zero, then that function is zero).

By considering the factorisation $(a - b\sqrt{2})(a + b\sqrt{2}) = a^2 - 2b^2$, show that the fractional part of $b\sqrt{2}$ is at least $c/|b|$, for some absolute constant $c > 0$.

Hence show that when $u_n = n\sqrt{2}$ for all n we have

$$S_N - \frac{N}{3} \leqslant C' N^{4/5},$$

where C' is another absolute constant.

SECTION B

3

Let $\alpha > 0$ and let $k \in \mathbb{N}$. Show that, for sufficiently large N, any set $A \subseteq \{1, \ldots, N\}$ with $|A| \geqslant \alpha N$ contains distinct $x_1, \ldots, x_k$ such that the average $\frac{1}{k}(x_1 + \cdots + x_k)$ also lies in A.

4

Suppose that $n \geqslant 2$ and that $f : \mathbb{F}_2^n \to \{0, 1\}$ is a boolean function. What is meant by the *influence* $I_k(f)$, where $1 \leqslant k \leqslant n$?

Determine $I_1(f)$ when

$$f(x_1, \ldots, x_n) = x_1 x_2 + \cdots + x_{n-1} x_n = \sum_{i<j} x_i x_j .$$

Give an example, without proof, of a function for which $\mathbb{P}(f = 0), \mathbb{P}(f = 1) \geqslant \frac{1}{4}$ such that $I_k(f) \leqslant 100 \log n / n$ for all k.

Show that for every boolean function f with $\mathbb{P}(f = 0), \mathbb{P}(f = 1) \geqslant \frac{1}{4}$ there is some k such that $I_k(f) \geqslant \log n / 100n$.

[You may assume Beckner's inequality without proof provided you state it clearly.]

SECTION C

5

Show that there is a compact subset of the plane containing a unit line segment in every direction, yet having measure zero. Discuss Fefferman's counterexample to the ball multiplier conjecture in $\mathbb{R}^2$, giving an overview of the argument and at least some of the more interesting technical details.

6

What is meant by the statement that a boolean function $f : \mathbb{F}_2^n \to \{0, 1\}$ may be computed using a circuit of depth d and size M? State the Hastad switching lemma. Give an overview of the proof of the following result due to Linial, Mansour and Nisan, including at least some of the more interesting technical details: if $f : \mathbb{F}_2^n \to \{0, 1\}$ is a boolean function computed using a circuit of depth d and size M then

$$\sum_{|r|>t} |\hat{f}(r)|^2 \leqslant CM2^{-ct^{1/d}},$$

where $|r|$ denotes the number of nonzero entries of r (or, equivalently, the Hamming distance of r from 0) and $c, C > 0$ are absolute constants.

END OF PAPER