UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 12 June, 2012 1:30 pm to 4:30 pm

PAPER 79

ADVANCED TOPICS IN FLUID MECHANICS OF CLIMATE

You may attempt ***ALL*** *questions, although full marks can be achieved by good answers to* ***THREE*** *questions.*
Completed answers are preferred to fragments.

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Rossby waves on a zonal jet

Rossby waves can be supported by variations in the rotation rate or bottom topography. In some cases, these waves can also be supported by a large-scale mean flow. To show this, consider a steady mean flow, $\overline{u} = U_0 sin\left(\frac{\pi y}{L}\right)$ in geostrophic balance. [*Assume that the Coriolis parameter $f = f_0$, and the bottom depth, $H = H_0$ are constant.*]

What is the corresponding free surface elevation, $\overline{\eta}$, in a uniform density fluid?

Start from the shallow water potential vorticity (PV) equation:

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0. \tag{1}$$

Let H_0 be the depth of the fluid at rest, and f_0 be the Coriolis parameter. Derive the quasi-geostrophic (QG) equation, including the mean flow given above. Identify the steady QG potential vorticity $\overline{q}$ associated with $\overline{u}$ and $\overline{\eta}$. By writing $q = q' + \overline{q}$, show that the equation for departures from the mean QG PV can be written

$$\frac{\partial q'}{\partial t} + J(\psi', q') + \overline{u}\frac{\partial q'}{\partial x} + v'\frac{\partial \overline{q}}{\partial y} = 0. \tag{2}$$

Linearize Eq. (2), and assuming that variations in $\overline{q}$ are on sufficiently large scale, derive the dispersion relation associated with Eq. (2) using plane-wave solutions of the form

$$\psi' = \hat{\psi} e^{i(kx+ly-\omega t)}. \tag{3}$$

where ψ' is the streamfunction associated with the perturbation velocity (u', v'). Show that the wavelength must be $< L/2$.

Stationary waves have a phase speed that exactly opposes the mean flow so that $\omega = 0$ in a stationary coordinate frame. Show that waves that are stationary with respect to the mean flow, $\overline{u}$ have a wavenumber $|\mathbf{k}| = \pi/L$ and a group velocity

$$\mathbf{c}_g = 2\overline{u}\frac{|\mathbf{k}|^2}{|\mathbf{k}|^2 + k_d^2}, \tag{4}$$

where $k_d^2 \equiv f_0^2/(gH_0)$. Qualitatively describe what this implies for waves generated as a mean flow passes over a stationary obstacle like a mountain range.

2 The atmospheric ocean

In most locations on the globe, land masses prevent the ocean from flowing uninterrupted eastward around the globe. An important exception is the Southern Ocean, where ocean dynamics more closely resemble those in the atmosphere. As a model for the circulation in the Southern Ocean, consider the following linearised equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla p - r\mathbf{u} + \frac{1}{\rho_0}\frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{1}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b, \tag{2}$$

$$\frac{\partial b}{\partial t} = \frac{\partial B}{\partial z} - \alpha b, \tag{3}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{4}$$

in the domain shown in Figure 1(a). The coordinates are aligned so that x points to the east, y points to the north, and z points up so that both gravity and f point in the negative z direction (consistent with the southern hemisphere). Here, r and α are Rayleigh friction and Newtonian cooling coefficients, $\boldsymbol{\tau}$ is the frictional stress, and B is the buoyancy flux.

Show that in the absence of friction and cooling, the steady-state, depth-integrated flow follows Sverdrup balance:

$$\beta\overline{v} = \nabla \times \frac{\boldsymbol{\tau}^w}{\rho_0}, \tag{5}$$

where the overbar denotes an integral over the full depth of the ocean, $\boldsymbol{\tau}^w$ is the wind stress, and $\beta = \partial f/\partial y$. State the assumptions used to derive this equation. A possibly useful vector identity is given at the end of the problem.

While Sverdrup balance is generally a good approximation in the relatively quiescent ocean gyres, in the Southern Ocean the frictional terms are important in setting the dynamics and result in a mean meridional circulation called the Deacon cell. Unlike the atmosphere, forcing in the ocean is concentrated near the sea surface. Suppose that the wind and buoyancy flux are only felt for depths shallower than some depth, h. Specifically, consider the following form for the wind and buoyancy forcing:

$$\boldsymbol{\tau}^w = \begin{cases} \tau_0^w \left(\frac{z+h}{h}\right)^2 \sin\left(\frac{\pi y}{L}\right)\hat{\boldsymbol{\imath}}, & -h < z < 0 \\ 0, & -H < z < -h \end{cases} \tag{6}$$

$$B = \begin{cases} B_0 \left(\frac{z+h}{h}\right) \cos\left(\frac{\pi y}{L}\right), & -h < z < 0 \\ 0, & -H < z < -h\,. \end{cases} \tag{7}$$

Using these forcing terms, look for steady solutions to Eqns. (1-4) in the zonal channel sketched in Figure 1(a). For simplicity, assume that the flow is independent of the x-direction, the bottom depth H is constant, and impose conditions of no normal flow at the walls. Write down the streamfunction associated with flow in the y-z plane and use arrows to indicate the directions of v and w on Figure 1(b), assuming that τ_0^w and B_0 are positive.

Typical values of the various length scales in the Southern Ocean are $H \simeq 1$km, $h \simeq 100$m, $L \simeq 1000$km, while the scaled wind and buoyancy forcing are typically of

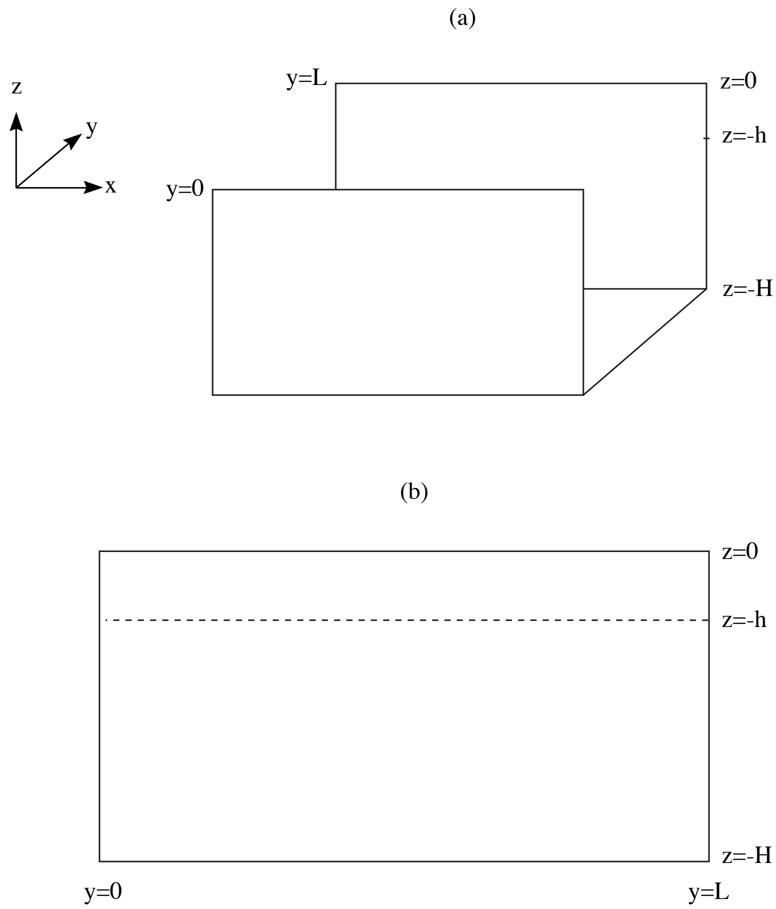

Figure 1: (a) Channel geometry (b) Roughly sketch the flow found in part (ii).

comparable magnitude: $f\tau_0/\rho_0 \simeq B_0$. If we further assume that $r \simeq \alpha$, show that the buoyancy forcing is negligible compared to the wind forcing. What extra dynamical feature, which is not explicitly considered in the model above, is needed to balance the wind-driven streamfunction? Qualitatively describe what happens to the energy input by the winds.

[*Hint: Vector identity:*

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}. \quad] \tag{8}$$

3

In a non-rotating system, unstratified fluid is flowing with velocity $U(z)$ over a horizontal plane, driven by an external pressure gradient P_x in the $x-$direction: the $z-$axis is vertical. Turbulence is generated by a stress τ_S acting on the fluid at the surface of the plane.

(i) Write down an equation for the vertical gradient of τ for $z > 0$ and find the height over which it is reasonable to assume that τ is constant. Using similarity theory, or otherwise, derive the form of the velocity profile $U(z)$ over this region where τ is independent of z. Discuss the differences between smooth and rough boundaries. Find also the form of the energy dissipation ϵ in terms of a velocity scale and z.

(ii) Buoyancy is introduced into the flow by adding or removing heat at the lower boundary.

(a) Show how the addition of a buoyancy flux B introduces an extra dimensionless parameter into the system, and write down the modified equation for $\frac{\mathrm{d}U}{\mathrm{d}z}$.

(b) In moderately stable and moderately unstable conditions determine approximate forms for $U(z)$ over a rough boundary, and describe the parameter range over which these forms are valid.

(c) Describe qualitatively the effect of buoyancy fluxes of either sign on the profile $U(z)$, and explain them physically.

(d) Define the gradient Richardson number and the flux Richardson number. Explain their physical significance, and find the relation between them

(iii) When the lower boundary is heated, explain what is meant by forced and free convection.

For the case of free convection, derive an expression for the buoyancy frequency N. Assuming Reynolds analogy is valid, use dimensional analysis to determine the form of the energy dissipation in free convection and contrast its form with that found for neutral flow.

(iv) Determine the velocity profile and buoyancy frequency in very stable conditions. Using the turbulent kinetic energy equation, or otherwise, determine the lengthscale at which dissipation occurs. What is the flux Richardson number in this case?

4

A two-dimensional, Boussinesq, turbulent plume rising up the underside of a glacial terminus begins to nucleate ice crystals whose buoyancy rapidly dominates the dynamics of the rising plume. Consider the rise of such a plume into an ocean whose salinity C_0 and temperature T_0 are equal to the liquidus temperature $T_L(C_0) = -mC_0$, where m is the constant liquidus slope. Steady equations governing the width $b(z)$ of the turbulent plume, the mean vertical velocity $w(z)$, mean temperature within the plume and ice crystals $T(z)$, bulk composition $\overline{C}(z)$, and volume fraction of crystals $\phi(z)$ are given by

$$\frac{d}{dz}(bw^2) = bg(\rho_0 - \overline{\rho})/\rho_0, \tag{1}$$

$$\frac{d}{dz}\left[\{c_p(T - T_0) - L\phi\}\, bw\right] = 0, \tag{2}$$

$$\frac{d}{dz}\left[\left(\overline{C} - C_0\right) bw\right] = 0, \tag{3}$$

where $\overline{\rho} = \phi\rho_s + (1 - \phi)\rho_0$ is the bulk density of the suspension, ρ_s is the density of ice, ρ_0 is the density of the ocean water, g is the acceleration due to gravity, c_p is the specific heat capacity, L is the specific latent heat and z is the vertical coordinate.

What physical laws do these equations represent? What boundary conditions should be applied? What physical law is needed to complete the description of the system? Show how this law can be expressed in terms of an approximation that relates the turbulent entrainment into the plume to the mean flow.

Using integral, or box-model arguments, derive equations (1) and (3).

It may be assumed that the turbulent plume is in thermodynamic equilibrium, so that

$$\overline{C} = (1 - \phi)C, \quad \text{where} \quad C = -T/m \tag{4}$$

is the salinity of the interstitial brine within the plume. Use these relationships to eliminate T and $\overline{C}$ from equations (2), (3) and (4), and find b, w and ϕ as functions of height and the reduced gravity for a self-similar plume which is valid when $\phi \ll 1$. Neglect all variations in density except when they modify the buoyancy.

END OF PAPER