UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 8 June, 2012 1:30 pm to 4:30 pm

PAPER 78

WAVE PROPAGATION AND SCATTERING

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Starting from the equations of mass and momentum conservation, derive the Ffowcs Williams–Hawkings integral equation for the special case of sound generated in a region $f(\boldsymbol{x}) > 0$ outside a fixed rigid impermeable object occupying $f(\boldsymbol{x}) < 0$. You may assume that, for any function $\phi(\boldsymbol{x}, t)$,

$$\int_{\mathbb{R}^3} \phi\, \delta(f) |\boldsymbol{\nabla} f| \, \mathrm{d}V = \int_{f=0} \phi \, \mathrm{d}S.$$

What is meant by an observer being "in the far field", and what is meant by a source being "compact"? How does the Ffowcs Williams–Hawkings equation derived above simplify for a compact source observed in the far field?

Stating explicitly any assumptions you make, compare the sound generated by (a) a region of turbulence and (b) the same region of turbulence containing a fixed rigid impermeable object, stating how the radiated sound power from the surface and volume integrals each scale with the Mach number of the turbulence. How big does the object need to be for (a) and (b) to give the same magnitude of sound? Briefly justify on physical grounds why the presence of the object should change the sound power radiated.

2

In a fluid of density $\rho_0 = 1$ and sound speed $c_0 = 1$, a thin elastic sheet of mass per unit area m is pinned at $(x, y) = (0, 0)$ and stretched along $x < 0, y = 0$ with a tension T, so that its displacement $\eta(x)$ in the y-direction is given by

$$m\frac{\partial^2 \eta}{\partial t^2} = T\frac{\partial^2 \eta}{\partial x^2} + P,$$

where P is the net pressure acting on the sheet. A plane wave is incident on the sheet with $\rho_{\text{inc}} = \exp\left\{\mathrm{i}\omega t + \mathrm{i}\omega(x\cos\theta_0 + y\sin\theta_0)\right\}$, giving a density perturbation $\rho' = \rho_{\text{inc}} + \phi$.

Show that the Wiener–Hopf equation for the Fourier transform $\tilde{\phi}(k, y)$ of $\phi(x, y)$ can be written as

$$K(k)\left[\tilde{\phi}\right]_{y=-0}^{y=+0} + \frac{\partial \Phi^+}{\partial y} = \frac{\omega \sin\theta_0}{k + \omega\cos\theta_0} + \frac{T\omega^2\eta'(0)}{m\omega^2 - Tk^2},$$

where $\Phi^+(k, y)$ is the right-half-range Fourier transform of $\phi(x, y)$, and $K(k)$ should be given explicitly. Assuming that $K(k)$ may be factorized as $K(k) = K^+(k)K^-(k)$ (which should not be found explicitly), for which values of k do K^+ and K^- have singularities? What is the strip $\mathcal{D}$ on which both K^+ and K^- are analytic and nonzero?

Assuming the entire function $E(k)$ involved in solving the Wiener–Hopf equation is identically zero, find $\phi(x, y)$ in integral form. Show that the integral form of $\eta(x)$ has a pole from $\partial\Phi^+/\partial y$ in the lower-half of the k-plane (you need not find this integral form). What does this pole physically represent, and how might it imply a value for the as yet undetermined constant $\eta'(0)$?

3

A time-harmonic wave $\psi(\mathbf{r})e^{-i\omega t}$ in three-dimensional space is incident upon an inhomogeneity with refractive index $n(\mathbf{r})$ which occupies a volume D in free space.

(a) Derive the first term of both the Born and the Rytov approximations for the space-dependent part of the total field at point $\mathbf{r}$, and denote them respectively by $\psi_B(\mathbf{r})$ and $\psi_R(\mathbf{r})$.

Show that the first-order term in a power series expansion of the Rytov approximation is equal to the Born approximation.

(b) In the case where the incident field is a monochromatic plane wave propagating with wave number k_0 in a direction $\mathbf{r}_0$, derive far-field approximations to $\psi_B(\mathbf{r})$ and $\psi_R(\mathbf{r})$.

(c) For the same incident wave as in (b), assume that the refractive index in D is $n(\mathbf{r}) = 1 + \mu W(\mathbf{r})$, where $\mu \ll 1$ and $W(\mathbf{r})$ is a stationary random function of position with Gaussian p.d.f. and mean zero, normalised so that $\langle W^2(\mathbf{r})\rangle = 1$.

The mean intensity in the Rytov approximation is given by

$$I(\mathbf{r}) = \langle \psi_R^*(\mathbf{r})\psi_R(\mathbf{r})\rangle \tag{1}$$

Derive an expression for $I(\mathbf{r})$ in the far field in terms of the autocorrelation function of the 'scattering potential' $V = k_0^2[n^2(\mathbf{r}) - 1]$.

[You may wish to use $\mathrm{Re}(f) = \frac{1}{2}(f + f^*)$ for a complex function f. You may also wish to use the Taylor expansion when calculating $\langle \exp(\phi)\rangle$ for a random phase ϕ, recalling that the random process in this question, the refractive index, has mean zero.]

4

Let $A : X \mapsto Y$ be a compact operator between 2 Hilbert spaces, and consider the problem $Ax = y$.

(a) Define a singular value system $\{\sigma_n; u_n; v_n\}$ for A, and use it to construct an expression for the Moore-Penrose generalised inverse $x^\dagger$.

(b) Define a regularisation strategy for the operator A. Given that a family of regularising operators R_α, with parameter $\alpha > 0$, is defined by

$$R_\alpha y = \sum_{n=1}^{\infty} f_\alpha(\sigma_n) \frac{(y, v_n)}{\sigma_n} u_n \ , \tag{1}$$

derive explicitly a function $f_\alpha(\sigma)$ such that the above regularisation strategy coincides with Tikhonov regularisation $(\alpha + A^* A)x_\alpha = A^* y$.

(c) Consider the operator $A : \mathrm{L}^2([0,1]) \mapsto \mathrm{L}^2([0,1])$, defined for all $f(x)$ in $\mathrm{L}^2([0,1])$ by

$$Af(x) = \int_0^x f(x')dx' \ . \tag{2}$$

(i) Check that $\{\sigma_n; u_n; v_n\}$ is a singular system for A, where

$$\sigma_n = \frac{2}{(2n-1)\pi} \quad , \quad u_n(x) = \sqrt{2} \cos \frac{x}{\sigma_n} \quad , \quad v_n(x) = \sqrt{2} \sin \frac{x}{\sigma_n} \ . \tag{3}$$

[*Hint: use the definition of the adjoint A^* given by $(Af, g) = (f, A^* g)$, with the usual inner product for square integrable functions.*]

(ii) Using Tikhonov regularisation, write explicitly the regularised solution x_α for this problem.

END OF PAPER