UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 9:00 am to 12:00 pm

PAPER 77

CONVECTION

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

Each question in Section I carries 10 marks,
each question in Section II carries 20 marks,
and the question in Section III carries 40 marks.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

Write down the dimensionless equations for thermal convection in a layer with stress-free and perfectly conducting boundaries, defining the dimensionless parameters (the Prandtl and Rayleigh numbers) that appear.

Give expressions in terms of these parameters for the exponential growth rate of infinitesimal disturbances with wavenumber k, when the Prandtl number is (a) very large and (b) equal to unity. In case (a) show that when $R \gg 1$ the maximum growth rate occurs for $k^2 \approx \pi^2$ and is approximately equal to $R/4\pi^2$.

2

The reduced model for weakly nonlinear thermosolutal convection takes the form

$$\begin{aligned}
\dot{a} &= \sigma(-a + rb - r_s d) \\
\dot{b} &= -b + a(1-c) \\
\dot{c} &= \varpi(-c + ab) \\
\dot{d} &= -\tau d + a(1-e) \\
\dot{e} &= \varpi(-\tau e + ad).
\end{aligned}$$

(i) Explain briefly the meaning of the variables $a, b, c, d, e,$ and the parameters $\sigma, \tau, r, r_s, \varpi$.

(ii) Show that on the steady solution branch r and a^2 are related by $r = (1 + a^2) + r_s\tau(1+a^2)/(\tau^2 + a^2)$. Show that if $\tau < 1$ and $r_s > \tau^3/(1-\tau)^2$ the steady solution branch is subcritical, and that in this case the minimum value r_{min} of r occurs when $a^2 = -\tau^2 + \sqrt{r_s\tau(1-\tau^2)}$, and takes the value

$$r_{min} = (\sqrt{r_s\tau} + \sqrt{1-\tau^2})^2$$

(iii) Now let β be defined by $r_s = \beta^2\tau$. It is given that for very small τ the critical value of r for oscillations, $r^{(0)}$, is approximately $1 + \tau\delta + \beta^2\tau/\delta$, where $\delta = \sigma/(1+\sigma)$. Show that, if terms of order τ^2 are neglected, r_{min} is never greater than $r^{(0)}$.

SECTION II

3

In mildly subcritical thermosolutal convection two-dimensional motion is described by the evolution equation for the complex amplitude $A(x,t)$:

$$\dot{A} = \mu A + \alpha|A|^2 A - |A|^4 A + A_{xx}, \quad \alpha > 0; \quad \mu, \alpha \text{ real}$$

(a) Find the range of μ in which uniform patterns (steady solutions independent of x) exist, and are stable to x-independent disturbances.

(b) Now consider the possibility of a steady solution in which $A \to 0$, $x \to \infty$, and $|A| \to$ const., $x \to -\infty$. Writing $A = R(x)e^{i\phi(x)}$, and considering the real and imaginary parts of the equation, show that for a solution of this type $d\phi/dx = 0$. Then consider the equation for the real variable R, and show that such a solution can only exist for a unique value of $\mu < 0$, which should be determined.

4

The equation for the vertically averaged temperature $\Theta(x,y,t)$ in long-wavelength convection with (slightly) asymmetric boundary conditions takes the form

$$\frac{\partial\Theta}{\partial t} = -\alpha\Theta - \mu\nabla^2\Theta - \nabla^4\Theta - \epsilon\gamma\nabla\cdot(\Theta\nabla\Theta) + \nabla\cdot(|\nabla\Theta|^2\nabla\Theta),$$

where γ is a constant and $\epsilon \ll 1$. Now assume that $\mu = \mu_c + \mu_2\epsilon^2$, where μ_c is the critical value of μ for instability for small disturbances. Writing $\partial/\partial t = \epsilon^2\partial/\partial T$, $\Theta = \epsilon(A(T)e^{i\mathbf{k}_1\cdot\mathbf{x}} + B(T)e^{i\mathbf{k}_2\cdot\mathbf{x}} + C(T)e^{i\mathbf{k}_3\cdot\mathbf{x}} + c.c) + \epsilon^3\Theta_3 + \ldots$, where $\mathbf{k}_1 = k_c(1,0)$, $\mathbf{k}_2 = k_c(-1/2, \sqrt{3}/2)$, $\mathbf{k}_3 = k_c(-1/2, -\sqrt{3}/2)$, and k_c is the optimum wavenumber, derive coupled evolution equations for the complex amplitudes A, B, C. Show that roll type solutions $(A \neq 0, B = C = 0)$ are never stable.

SECTION III

5

Convection in a layer heated from below in the presence of an imposed vertical magnetic field is governed by the usual temperature equation, together with the induction equation for the solenoidal magnetic field $\mathbf{B}$:

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B},$$

where η is the magnetic diffusivity. There is also an additional body force $\mu_0^{-1} \mathbf{B} \cdot \nabla \mathbf{B}$ (the Lorentz force) on the right hand side of the momentum equation.

(i) Writing $\mathbf{B} = B_0(\hat{\mathbf{z}} + \mathbf{b})$, show that after nondimensionalization the linearised momentum and induction equations take the form

$$\begin{aligned} \frac{1}{\sigma} \dot{\mathbf{u}} &= -\nabla P + Q\zeta \frac{\partial \mathbf{b}}{\partial z} + R\theta \hat{\mathbf{z}} + \nabla^2 \mathbf{u}, \\ \dot{\mathbf{b}} &= \frac{\partial \mathbf{u}}{\partial z} + \zeta \nabla^2 \mathbf{b}, \end{aligned}$$

where $\zeta = \eta/\kappa$, $Q = B_0^2 d^2/\mu_0 \rho_0 \eta \nu$.

(ii) Assuming perfectly conducting, stress-free boundaries, and that $\mathbf{b}$ is vertical at the upper and lower boundary, show that solutions can be found in which $b_z \propto \cos \pi z$, $b_{x,y} \propto \sin \pi z$, and hence derive the dispersion relation for the growth rate λ of disturbances with horizontal wavenumber k. Show that the condition for marginal steady solutions is $R = R^{(e)} = ((\pi^2 + k^2)^3 + Q\pi^2(\pi^2 + k^2))/k^2$, and find the condition $R = R^{(o)}$ for marginal oscillatory convection. Show that marginal oscillations can exist only when $\zeta < 1$ and Q is sufficiently large.

(iii) Now consider the case $Q \gg 1$. Show that for both steady and oscillatory convection R is minimised when $k \sim Q^{1/6}$, and find both the critical wavenumber and the minimum value of R correct to order $Q^{2/3}$.

For general Q, derive the explicit expression for Q in terms of the minimum value R_c of $R^{(e)}(k)$ for steady convection and $R_0 = 27\pi^4/4$, the critical value when $Q = 0$:

$$Q\pi^2 = R_c - R_c^{\frac{2}{3}} R_0^{\frac{1}{3}},$$

and give an expression for the corresponding value of k^2 in terms of R_c and R_0.

END OF PAPER