UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 31 May, 2012 1:30 pm to 4:30 pm

PAPER 76

PERTURBATION AND STABILITY METHODS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Distinguish between a *regular* and a *singular* perturbation. Illustrate your answer by finding two terms of an asymptotic expansion as $\epsilon \to 0$ for the roots of the equation

$$\epsilon x^3 + x^2 - 2 = 0 \ .$$

(b) The function $f(z)$ is defined as

$$f(z) = \int_0^z \exp(-t^2)\mathrm{d}t \ .$$

Obtain the leading order term of an asymptotic expansion (or expansions) of $f(z)$ for $|z| \to \infty$ and fixed $\arg(z)$ in the sector $0 \leqslant \arg(z) \leqslant \pi/2$.

(c) The (Bessel) function $g(x)$ is defined for real $x > 0$ as

$$g(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\mathrm{d}t \ .$$

Obtain the leading order term of an asymptotic expansion for $g(x)$ in the limits $x \to 0$ and $x \to \infty$. [*If you quote a standard result, a brief explanation should be given.*]

2

Consider the WKB equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + k^2 y = 0 \, ,$$

where k is a real function of $X = \epsilon x$. Show, without approximation, that the equation has solutions of the form

$$y = A(X,\epsilon) \exp\left[\mathrm{i} \int^X \sigma(X',\epsilon) \mathrm{d}X'/\epsilon\right] \, ,$$

where A and σ are real, provided that σA^2 is constant and A satisfies a differential equation to be determined.

Now suppose that $\epsilon \to 0$. By expanding both A and σ, obtain leading order approximations σ_0, A_0 and hence y_0. Obtain also an expression involving k for the first correction σ_1.

Consider now the Sturm-Liouville problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \lambda(x+1)^2 y = 0 \qquad \text{with } y(0) = y(1) = 0.$$

Use the results above to determine the eigenvalues λ_n as $\lambda_n \to \infty$, finding two terms of an asymptotic expansion for λ_n.

3

(a) The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + Axy' + xy = 0 \qquad \text{with } y(0) = y(1) = 1 \ ,$$

where A is a constant and $\epsilon \ll 1$. In the case $A > 0$ determine the location and size of the inner region, and find the first non-zero terms in the outer and inner regions. Without detailed calculation, explain what happens when $A < 0$ and when $A = 0$.

(b) In the following, the Fourier Transform (FT) is defined by

$$\overline{f}(k) = \int_{-\infty}^{\infty} \exp(-\mathrm{i}kx) f(x) \mathrm{d}x \ .$$

Show that:

(i) $\overline{\mathrm{H}(x)} = \pi\delta(k) + \frac{1}{\mathrm{i}k}$;

(ii) $\overline{1/x} = -\mathrm{i}\pi \mathrm{sgn}(k)$;

(iii) $\overline{\log|x|} = -(\pi/|k|) + C\delta(k)$,

where the constant C need not be determined.

Consider the periodic real generalised function defined to be $f(x)$ in $0 \leqslant x \leqslant 1$ and elsewhere by periodicity. Show that, if we write

$$f(x) = \sum_{m=-\infty}^{\infty} c_m \exp(2\pi\mathrm{i}mx) \ ,$$

then the c_m can be written in terms of the FT of $f(x)[\mathrm{H}(x) - \mathrm{H}(x-1)]$. Find the behaviour of c_m as $m \to \infty$ when $f(x) = \log|x - 1/2|$.

4

Consider the linearised Ginzburg-Landau (GL) equation

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} - \mu\eta - \gamma\frac{\partial^2 \eta}{\partial x^2} = F(x,t) \ , \tag{1}$$

where $F(x,t)$ is a given forcing function and U and γ are real positive constants.

(a) Consider the case in which μ is a positive constant and $F(x,t) = \delta(x)\delta(t)$. By taking a double Fourier transform of equation (1), or otherwise, show that the Green's function of the GL equation is

$$\frac{1}{2\sqrt{\pi\gamma t}} \exp\left\{\mu t - \frac{(x-Ut)^2}{4\gamma t}\right\} \mathrm{H}(t) \ . \tag{2}$$

Hence, describe the different possible behaviours of solutions of the GL equation in the limit $t \to \infty$.

(b) Now suppose that $F(x,t) \equiv 0$ and $\mu = \mu_0 - \nu\epsilon^2 x^2$ with $\epsilon \ll 1$ and $\nu > 0$, and write

$$\eta(x,t) = \exp(Ux/2\gamma) b(\xi) \exp(-\mathrm{i}\omega_0 t - \mathrm{i}\epsilon\omega_1 t) \ .$$

Here the function $b(\xi)$ satisfies

$$b'' + (\lambda - \xi^2) b = 0 \ , \tag{3}$$

with $\xi = \sqrt{\epsilon} f x$ and λ, f are suitable constants (to be be determined). Find ω_0 and ω_1, and hence find a condition on μ_0 for the system to be globally unstable. How does this result relate to your answer to part (a)? [*Solutions of equation (3) which decay at infinity only exist when λ is a positive odd integer.*]

END OF PAPER