UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 9:00 am to 12:00 pm

PAPER 75

SLOW VISCOUS FLOW

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the Reciprocal Theorem for Stokes flow. Derive the Faxén relation

$$\mathbf{U} = \frac{\mathbf{F}}{6\pi\mu} + \left(1 + \frac{a^2}{6}\nabla^2\right)\mathbf{u}_\infty$$

for the velocity $\mathbf{U}$ of a rigid sphere of radius a that is placed into an unbounded Stokes flow $\mathbf{u}_\infty(\mathbf{x})$ and has a force $\mathbf{F}$ applied to it.

[*You should assume from the outset that the stress exerted on the surface of a rigid sphere translating with velocity* $\mathbf{V}$ *through unbounded fluid otherwise at rest is given by* $-3\mu\mathbf{V}/2a$.]

(b) State the Papkovich–Neuber representation for the velocity and pressure in Stokes flow.

Use this representation, explaining your choice of trial harmonic potentials, to determine the perturbation flow induced by a stationary rigid sphere placed at the origin of an unbounded uniform strain $\mathbf{u} = \mathbf{E}\cdot\mathbf{x}$. Why is the perturbation flow induced by a force-free, couple-free sphere placed in a general linear flow $\mathbf{u} = \mathbf{U}_0 + \mathbf{\Omega}\wedge\mathbf{x} + \mathbf{E}\cdot\mathbf{x}$ exactly the same?

(c) Two force-free, couple-free spheres of radius a are placed symmetrically about the origin of a uniform shear flow $\mathbf{u}_\infty = (Cy, 0, 0)$, where $C > 0$, such that the centres of the spheres are at $\pm(X(t), Y(t), 0)$ in Cartesian coordinates. As $t \to -\infty$, it is given that $Y(t) \to Y_\infty$ (and $X \to -\infty$), where $Y_\infty \gg a$.

Explaining any approximations, show that

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{5Ca^3}{8}\frac{XY^2}{R^5} + O(Ca^5/R^4),$$

where $R^2 = X^2 + Y^2$, and hence that the maximum value Y_m attained by $Y(t)$ is given by

$$Y_m = Y_\infty\left(1 + \frac{5a^3}{24Y_\infty^3} + O(a^5/Y_\infty^5)\right).$$

What happens to $Y(t)$ as $X \to \infty$, and why?

2

The rupture of a thin sheet of viscous fluid due to an attractive (van der Waals) force per unit area between the two surfaces of the sheet can be analysed in a similar manner to the Rayleigh instability:

Let the sheet occupy the region $-h(x,t) \leqslant y \leqslant h(x,t)$. Assume that the effect of the attraction is simply to modify the usual stress boundary condition on each interface to

$$[\boldsymbol{\sigma} \cdot \mathbf{n}]_-^+ = \left(\frac{V}{h^3} + \gamma\kappa\right)\mathbf{n},$$

where $V > 0$ is a constant coefficient of attraction, γ is the constant coefficient of surface tension, $\mathbf{n}$ is the normal to the interface and κ is its curvature. Gravity and the surrounding air should be neglected.

Consider small symmetric perturbations to a uniform thickness h_0 such that $h = h_0 + \eta(x,t)$, where $\eta \propto \exp(\mathrm{i}kx + st)$, $|\eta| \ll h_0$ and $|\eta_x| \ll 1$. Obtain the linearized boundary conditions at $y = h_0$. Using Papkovich–Neuber potentials χ and $\mathbf{\Phi} = (0, \phi)$ with the appropriate symmetry, deduce that the growth rate of the rupture instability is given by

$$s = \frac{3V}{\mu h_0^3} \frac{(1 - \Gamma K^2)\sinh^2 K}{K(2K + \sinh 2K)}\ , \tag{1}$$

where $K = kh_0$, and identify the constant Γ.

Sketch the form of $s(K)$ for the cases $\Gamma = 0$ and $\Gamma = 1$, and comment on the physical interpretation of the long and short wavelength behaviour. What additional physical effects might modify your prediction of the most unstable wavelength?

The surfaces of a planar soap film are uniformly covered with surfactant, which reduces the unperturbed coefficient of surface tension to a value γ_0. Use physical arguments, with diagrams, to explain why the surfactant decreases the growth rate of the rupture instability relative to that of a surfactant-free sheet with constant surface tension γ_0. [*Mathematical analysis is* **not** *required.*]

In the limit of strong surfactant effects, (1) is replaced by

$$s = \frac{3V}{\mu h_0^3} \frac{(1 - \Gamma K^2)(\sinh 2K - 2K)}{4K\cosh^2 K}\ .$$

Find the maximum growth rate for the case $\Gamma \gg 1$. Comment on the lifetime of a soap bubble.

3

A straight, vertical, axisymmetric thread of viscous fluid of density ρ and viscosity μ has cross-sectional area $A(z,t)$. Gravity acts in the positive z-direction and surface tension is negligible. The thread is surrounded by inviscid fluid of density ρ_a in which there is a hydrostatic pressure gradient $p_a(z) = p_0 + \rho_a g z$.

Assuming that $\partial A/\partial z \ll A^{1/2}$, derive the equation

$$\frac{3\mu}{A}\frac{\partial}{\partial z}\left(A\frac{\partial w}{\partial z}\right) + (\rho - \rho_a)g = 0\ ,$$

where w is the vertical velocity, explaining your argument carefully. Obtain another relationship between A and w.

(a) For a steady flow with vertical flux Q, show that

$$\frac{1}{2}\left(\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 = \frac{U}{w} + c,$$

where U is to be identified and c is a constant.

(b) Now let $\rho = \rho_a$ (or, equivalently, suppose that gravity is negligible). At $t = 0$ the thread has length $2L_0$ and cross-sectional area $A_0(z)$. For $t > 0$ the thread is stretched by pulling on the ends of the thread at $z = \pm L(t)$ with equal and opposite forces $\pm F(t)$.

By considering the evolution of a fluid element, show that

$$A(z,t) = A_0(z_0) - \Delta(t),$$

where $\Delta(t) = \dfrac{1}{3\mu}\displaystyle\int_0^t F(t')\,\mathrm{d}t'$ and $z = z_0 + \displaystyle\int_0^{z_0} \frac{\Delta(t)\,\mathrm{d}z_0'}{A_0(z_0') - \Delta(t)}$. Assume that the fluid initially at $z = 0$ remains there.

Find L as a function of Δ when $A_0(z) = C(1 + k^2 z^2)$, where C and k are constants. If F is constant show that L becomes infinite after a finite time t^*, and determine t^*. Show also that $L \sim B(t^* - t)^\beta$ as $t \to t^*$, and find the constants B and β for the two cases $k \neq 0$ and $k = 0$. Comment briefly on why the values of β differ between the cases and explain why B does not depend on L_0 for $k \neq 0$.

[*You may assume that* $\displaystyle\int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.]

4

A rigid circular cylinder of radius a falls through viscous fluid with its axis horizontal and parallel to a rigid vertical wall. With respect to suitably chosen coordinates (x, y), the wall lies along $y = 0$ and has velocity $(-U, 0)$, the axis of the cylinder is at $(0, (1+\epsilon)a)$ and is stationary, the cylinder rotates about its axis with angular velocity Ω, and the fluid occupies the region outside the cylinder in $y > 0$. Assume throughout that $0 < \epsilon \ll 1$.

Use the lubrication approximation to determine the flow in the thin gap between the cylinder and the wall. Show that the flux q through the gap is $\frac{2}{3}\epsilon a(\Omega a - U)$ (per unit axial length of cylinder).

Show also that

$$\frac{\sigma_{xy}}{\mu}\bigg|_{y=0} = \frac{4U - 2\Omega a}{h} + \frac{6q}{h^2},$$

where $h(x)$ is the gap width, and find a similar expression for σ_{xy} on $y = h$. Hence calculate the tangential force (per unit length of cylinder) that is exerted by the shear stress in the thin gap (i) on the wall and (ii) on the cylinder. Why are these forces not equal and opposite?

The cylinder has a uniform density that is $\Delta\rho$ greater than that of the fluid. What is the force and couple on the cylinder? Use your answers to (i) and (ii) to find U and to show that $\Omega = 0$ at leading order.

For $\Omega = 0$, sketch, as functions of x, the pressure $p(x)$ and the shear stress $\sigma_{xy}(x, h)$ on the cylinder. [*You need not find p explicitly.*] Identify the two points where the shear stress vanishes on the cylinder. Sketch the streamlines of the flow in the thin gap.

Now consider a two-dimensional Couette flow with rigid walls at $y = \pm a(1+\epsilon)$ which have velocity $(\pm U, 0)$. A force-free, couple-free, rigid cylinder of radius a is introduced perpendicular to the flow with its axis initially located at $(\lambda\epsilon a, 0)$, where $-1 < \lambda < 1$. Adapt your previous results to deduce the speed V of the cylinder. For $\lambda = 0$ sketch the streamlines on either side of the cylinder, and explain why its angular velocity is $O(\epsilon^{1/2}U/a)$.

[*You may assume that if* $I_n \equiv \displaystyle\int_{-\infty}^{\infty} \frac{d\xi}{(1+\xi^2)^n}$ *then* $I_1 = \pi$, $I_2 = \dfrac{\pi}{2}$ *and* $I_3 = \dfrac{3\pi}{8}$.]

END OF PAPER