UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2012 1:30 pm to 4:30 pm

PAPER 74

BIOLOGICAL PHYSICS

*Attempt **ALL** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A microsphere of radius a and drag coefficient ζ is constrained to move along the x-axis, and is acted on by an optical trap which is moving in the positive x-direction at velocity v_T. When the trap is located at a point x_0 it exerts a force $F(x - x_0)$, so the overdamped dynamics of the particle is

$$\zeta \dot{x} = F(x - v_T t) \ .$$

Suppose that the trap has compact support, so that $F(x) = 0$ for $x < -X_L$ and for $x > X_R$. If the trap starts to the left of the particle, find the particle's net displacement Δx after the trap has passed it by, and the time Δt spent by the particle interacting with the trap. What is the condition that assures that the particle does not remain trapped as $t \to \infty$? Assuming this is the case, show that whatever the form of $F(y)$ the net displacement is always in the direction of the trap motion, and suggest a heuristic explanation for this result. Find the asymptotic behaviour of Δx for large trap velocities.

The trap is now moved around a circle of radius $R \gg a$. Derive the particle's net rotational frequency f_p as a function of the trap angular frequency $f_T = v_T/(2\pi R)$, the displacement Δx in each kick, the interaction time Δt and the potential width $2X_0 = X_R - X_L$. Confirm that in the regime of suitably large trap velocity, which you should define precisely, one obtains the intuitive result $f_p \simeq (\Delta x / 2\pi R) f_T$. Specializing to the case of a triangular trapping potential, with $F(x) = F$ for $-X_0 < x < 0$ and $F(x) = -F$ for $0 < x < X_0$, obtain an explicit expression for f_p / f_c as a function of the two quantities $\alpha = X_0/(\pi R)$ and $\beta = f_T/f_c$, where $2\pi R f_c = F/\zeta$.

2

An elastic filament with bending modulus A and length L has small-amplitude excursions $h(x)$ from the x-axis, and is characterized by the bending energy

$$\mathcal{E} = \frac{1}{2}\int_0^L dx A h_{xx}^2 \ .$$

a) Assuming that the filament is free, with no forces and torques acting anywhere on it, find the Euler-Lagrange equation and "natural" boundary conditions for $h(x)$. Show that these render the Euler-Lagrange operator self-adjoint.

b) From general principles we know that the set of eigenfunctions of such an operator define a complete set of basis functions. Show that these can be written as

$$W^{(n)}(x) = A\cos(k^{(n)}x) + B\sin(k^{(n)}x) + D\cosh(k^{(n)}x) + E\sinh(k^{(n)}x) \ ,$$

and find the transcendental equation satisfied by $k^{(n)}$. By a graphical construction or otherwise give approximate values for the infinite sequence of wave vectors $k^{(n)}$.

c) Use the principle of equipartition to find the variance of $h(x)$, using the expansion $h(x) = \sum a_n W^{(n)}(x)$.

d) Suppose the filament obeys the dynamical equation

$$\zeta h_t = -\frac{\delta \mathcal{E}}{\delta h} \ ,$$

where ζ is an appropriate drag coefficient. If at time $t = 0$ the filament has the shape $H(x)$, find its subsequent time evolution.

3

The Brusselator is a model reaction-diffusion system in which two chemical concentrations u and v evolve according to the coupled PDEs

$$u_t = D_u u_{xx} + a - (b+1)u + u^2 v \ , \qquad (1)$$

$$v_t = D_v v_{xx} + bu - u^2 v \ , \qquad (2)$$

where the parameters a, b, D_u, and D_v are positive constants. We shall consider all to be fixed except b, which serves as a control parameter.

a) Find the homogeneous steady state with u and v both positive. Linearization of the governing equations around this value leads to a 2×2 stability matrix A with entries a_{ij}. Explain the general conditions on $\mathrm{tr}A$ and $\det A$ for stability. Compute these conditions for the Brusselator.

b) Find the general criterion for stability of the system with diffusion and the wavenumber of the mode that appears first when a Turing instability occurs. Compute these for the Brusselator and thereby determine the critical value of b for the instability and the wavenumber k at onset.

END OF PAPER