UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 9:00 am to 11:00 am

PAPER 73

TURBULENCE

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

A data sheet of three pages
is attached

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Burgers' vortex consists of a vortex tube with vorticity $\boldsymbol{\omega} = (0, 0, \omega_z)$ and

$$\omega_z = \frac{\Gamma_0}{\pi\delta^2} \exp\left[-r^2/\delta^2 \right], \quad \Gamma_0 = \text{constant}\,,$$

which sits in the irrotational straining flow $u_r = -\frac{1}{2}\alpha r$, $u_z = \alpha z$ in cylindrical polar coordinates (r, θ, z). Here α is a positive constant which is a measure of the strain rate. Show that this is a solution of the unsteady vorticity equation provided that the characteristic radius, δ, satisfies

$$\frac{d\delta^2}{dt} + \alpha\delta^2 = 4\nu\,,$$

where ν is the kinematic viscosity. Confirm that δ tends to the steady solution $\delta = \sqrt{4\nu/\alpha}$ as $t \to \infty$, irrespective of its initial value. Give a physical interpretation of this result.

The azimuthal velocity in Burgers' vortex is $u_\theta = (\Gamma/2\pi r)\left[1 - \exp(-r^2/\delta^2)\right]$. Use order-of-magnitude arguments to show that, for $\Gamma_0/\nu \to \infty$, the steady solution has the property that the viscous dissipation per unit length of the tube is finite and independent of ν. Why are Burgers' vortex tubes good candidates for the centres of intense dissipation in a high Reynolds number turbulent flow, whereas Burgers' vortex sheets are not?

(ii) Starting with the vorticity equation, derive the governing evolution equation for the mean enstrophy, $\frac{1}{2}\langle \boldsymbol{\omega}^2 \rangle$, in homogeneous turbulence and use order-of-magnitude arguments to show that

$$\langle \omega_i \omega_j S_{ij} \rangle = \nu \langle (\nabla \times \boldsymbol{\omega})^2 \rangle \left[1 + O(\mathrm{Re}^{-1/2}) \right]\,,$$

where S_{ij} is the rate-of-strain tensor and $\mathrm{Re} = u\ell/\nu$ the Reynolds number based on the integral scales.

(iii) In any flow the third invariant, R, of the velocity gradient tensor, $A_{ij} = \partial u_i/\partial x_j$, can be written in terms of S_{ij} and $\boldsymbol{\omega}$ as

$$R = \frac{1}{3} A_{ij} A_{jk} A_{ki} = \frac{1}{3} S_{ij} S_{jk} S_{ki} + \frac{1}{4} \omega_i \omega_j S_{ij}\,.$$

Noting that R can be expressed as a divergence, show that, in homogeneous turbulence,

$$\langle \omega_i \omega_j S_{ij} \rangle = -\frac{4}{3} \langle a^3 + b^3 + c^3 \rangle\,,$$

where a, b and c are the principal rates of strain at any location. Hence confirm that

$$\langle \omega_i \omega_j S_{ij} \rangle = -4 \langle abc \rangle\,.$$

(iv) Use the results of (iii) to explain why the skewness of $\partial u_x/\partial x$ is negative and why bi-axial strain is more common than axial strain in homogeneous turbulence. What is the significance of the dominance of bi-axial strain?

(v) Briefly discuss the Townsend-Betchov cartoon of the energy cascade, distinguishing between the inertial-range and dissipative scales. Why does this cartoon, if substantially correct, pose a problem for certain mathematical models of the cascade?

2

(i) Explain what is meant by internal intermittency. Describe briefly the differences between intermittency at the integral scale and in the equilibrium range.

(ii) Landau used the existence of integral-scale intermittency to criticise Kolmogorov's inertial-range prediction

$$\langle(\Delta v)^p\rangle(r) = \beta_p \epsilon^{p/3} r^{p/3}\,, \quad \eta \ll r \ll \ell\,,$$

where $\langle(\Delta v)^p\rangle(r)$ is the usual structure function of order p, ϵ the average energy dissipation rate per unit mass, and the pre-factors β_p are assumed universal. Discuss the counter example provided by Landau and show how this counter example demonstrates that the β_p cannot be universal.

(iii) Kolmogorov refined his original 1941 theory to allow for equilibrium range intermittency. Discuss Kolmogorov's refined similarity hypothesis of 1962, explaining the role played by the random variable $\epsilon_{AV}(r)$, the dissipation rate averaged over the scale r at any location in the flow. What is the physical motivation for the refined similarity hypothesis and in what sense is universality retained in the modified theory?

Describe briefly how Kolmogorov used the empirical estimate $\langle\epsilon_{AV}^2(r)\rangle/\epsilon^2 = B(\ell/r)^\mu$, $\eta < r < \ell$, where B and μ are constants, in combination with the refined similarity hypothesis, to predict the exponents ζ_p in the refined inertial-range scaling $\langle(\Delta v)^p\rangle(r) \sim r^{\zeta_p}$. What aspect of this modified theory has attracted most criticism?

(iv) In both the original and modified theories of the intertial range, Kolmogorov assumed that $\langle(\Delta v)^2\rangle(r)$ filters out contributions to $\frac{1}{2}\langle\mathbf{u}^2\rangle$ that arise from eddies much larger than the scale r. Explain the usual rationalisation of the common, if imperfect, estimate

$$\frac{3}{4}\langle(\Delta v)^2\rangle(r) \approx \int_{\pi/r}^{\infty} E(k)dk\,.$$

[You may assume isotropy.] Discuss why a better, but still imperfect, estimate in isotropic turbulence is

$$\frac{3}{4}\langle(\Delta v)^2\rangle(r) \approx \int_{\pi/r}^{\infty} E(k)dk + \frac{r^2}{10}\int_0^{\pi/r} k^2 E(k)dk\,.$$

Evidently $\langle(\Delta v)^2\rangle(r)$ retains information about scales greater than r. Explain why this is also true for higher-order structure functions. Why is this a potential problem for Kolmogorov's theories?

(v) In two-dimensional turbulence a similar approximate expression relates $\langle(\Delta v)^2\rangle(r)$ to $E(k)$:

$$\frac{1}{2}\langle(\Delta v)^2\rangle(r) \approx \int_{\pi/r}^{\infty} E(k)dk + \frac{r^2}{8}\int_0^{\pi/r} k^2 E(k)dk\,.$$

However, in two dimensions the inertial-range scaling now takes the form $E(k) \sim k^{-3}$. Show that, if $E(k)$ exhibits a wide interial range, $\langle(\Delta v)^2\rangle(r)$ in the inertial range is now dominated by large-scale enstrophy, rather than by the energy below scale r.

3

(i) A flow consists of a spatially localised distribution of vorticity, $\boldsymbol{\omega}(\mathbf{x},t)$, with the far-field pressure and velocity both falling off as $|\mathbf{x}|^{-3}$ at large $|\mathbf{x}|$. Show that the linear impulse, $\mathbf{L} = \frac{1}{2}\int_{V_\infty} \mathbf{x} \times \boldsymbol{\omega} dV$, is an invariant of the vorticity field. You may need the identity

$$\frac{\partial}{\partial x_j}(F_i x_j) = \left[2\mathbf{F} + \nabla(\mathbf{x}\cdot\mathbf{F}) - \mathbf{x}\times(\nabla\times\mathbf{F})\right]_i ,$$

which holds for any smooth vector field $\mathbf{F}$. [The symbol V_∞ indicates an integral over all space.]

Let V_R be a large spherical control volume of radius R which encloses $\boldsymbol{\omega}(\mathbf{x},t)$. Then it may be shown that $\mathbf{L} = \frac{3}{2}\int_{V_R} \mathbf{u} dV$. Use the fact that both pressure and velocity fall off as $|\mathbf{x}|^{-3}$ at large $|\mathbf{x}|$ to explain the physical basis for the invariance of $\mathbf{L}$.

(ii) Show that, in isotropic turbulence, the energy spectrum at small k takes the form

$$E(k) = \frac{Lk^2}{4\pi^2} + \ldots , \quad L = \int_{V_\infty} \langle \mathbf{u}\cdot\mathbf{u}'\rangle d\mathbf{r} .$$

Show also that Saffman's integral, L, can be written as

$$L = \frac{\left\langle \left(\int_V \mathbf{u} dV\right)^2 \right\rangle}{V}$$

for some large control volume, V, embedded in the turbulence.

(iii) In the case of V being spherical, $V = V_R$, give a physical interpretation of the scaling

$$\left\langle \left(\int_{V_R} \mathbf{u} dV\right)^2 \right\rangle \sim V_R$$

in terms of the linear impulse of the eddies in V_R.

(iv) Show that, in isotropic turbulence, $L = 4\pi\left[r^3 u^2 f(r)\right]_{r\to\infty}$ where $f(r)$ is the usual longitudinal correlation function. What is the physical origin of the long-range correlation $f(r\to\infty) \sim r^{-3}$?

(v) Starting with the Karman–Howarth equation,

$$\frac{\partial}{\partial t}\langle \mathbf{u}\cdot\mathbf{u}'\rangle = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\left(r^4 u^3 K\right) + 2\nu\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\langle \mathbf{u}\cdot\mathbf{u}'\rangle ,$$

show that L is an invariant. [You may assume that the longitudinal triple correlation at large separation falls off as $K(r\to\infty) = O(r^{-4})$, or faster.] Use momentum conservation to provide a physical interpretation for the invariance of L.

(vi) Show that, when L is non-zero and the Reynolds number is large, the kinetic energy decays as $\frac{1}{2}\langle \mathbf{u}^2\rangle \sim t^{-6/5}$.

END OF PAPER

Mathematical Tripos Part III — Dr P. A. Davidson

Hydrodynamic Turbulence — Lent 2012

DATA SHEET: Kinematic Relationships in Isotropic Turbulence

Second-Order Velocity Correlations

$$\langle u_x(\boldsymbol{x})u_x(\boldsymbol{x}+r\hat{\boldsymbol{e}}_x)\rangle = u^2 f(r)\ , \qquad u^2 = \left\langle u_x^2\right\rangle = \left\langle u_y^2\right\rangle = \left\langle u_z^2\right\rangle$$

$$\langle u_y(\boldsymbol{x})u_y(\boldsymbol{x}+r\hat{\boldsymbol{e}}_x)\rangle = u^2 g(r) = \frac{u^2}{2r}\frac{d}{dr}(r^2 f)$$

$$\langle \boldsymbol{u}(\boldsymbol{x})\cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})\rangle = \frac{u^2}{r^2}\frac{d}{dr}(r^3 f)\ , \qquad r = |\boldsymbol{r}|$$

$$Q_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r})\rangle = \frac{u^2}{2r}\left[(r^2 f)'\delta_{ij} - f' r_i r_j\right]$$

Dissipation

$$\epsilon = \langle 2\nu S_{ij}S_{ij}\rangle = \nu\left\langle \boldsymbol{\omega}^2\right\rangle \sim u^3/l$$

Integral Scale, l

$$l = \int_0^\infty f(r)dr = \frac{1}{2u^2}\int_0^\infty \langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle\, dr$$

Taylor Microscale, λ

$$f = 1 - \frac{r^2}{2\lambda^2} + 0(r^4)$$

$$\lambda/l \approx \sqrt{15}(ul/\nu)^{-1/2}$$

$$\lambda^2 = \frac{15u^2}{\langle\boldsymbol{\omega}^2\rangle} = \frac{u^2}{\langle(\partial u_x/\partial x)^2\rangle}$$

Kolmogorov Microscales, η, v

$$\eta = (\nu^3/\epsilon)^{1/4} \approx l(ul/\nu)^{-3/4}$$

$$v = (\nu\epsilon)^{1/4} \approx u(ul/\nu)^{-1/4}$$

Structure Functions

$$\langle (\triangle v)^p = \langle (u_x(\boldsymbol{x} + r\hat{\boldsymbol{e}}_x) - u_x(\boldsymbol{x}))^p \rangle$$

$$\left\langle (\triangle v)^2 \right\rangle = 2u^2(1-f)$$

$$\left\langle (\triangle v)^3 \right\rangle = 6 \left\langle u_x^2(\boldsymbol{x}) u_x(\boldsymbol{x} + r\hat{\boldsymbol{e}}_x) \right\rangle$$

Third-Order Velocity Correlations

$$u^3 K(r) = \left\langle u_x^2(\boldsymbol{x}) u_x(\boldsymbol{x} + r\hat{\boldsymbol{e}}_x) \right\rangle$$

$$\begin{aligned} S_{ijk}(\boldsymbol{r}) &= \langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}) u_k(\boldsymbol{x} + \boldsymbol{r}) \rangle \\ &= u^3 \left[\frac{K - rK'}{2r^3} r_i r_j r_k + \frac{2K + rK'}{4r} \left(r_i \delta_{jk} + r_j \delta_{ik} \right) - \frac{K}{2r} r_k \delta_{ij} \right] \end{aligned}$$

Energy Spectrum

$$E(k) = \frac{1}{\pi} \int_0^\infty \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, kr \sin(kr) dr$$

$$\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle = 2 \int_0^\infty E(k) \frac{\sin(kr)}{kr} dk$$

$$\frac{1}{2} \left\langle \boldsymbol{u}^2 \right\rangle = \int_0^\infty E(k) dk$$

$$\frac{1}{2} \left\langle \boldsymbol{\omega}^2 \right\rangle = \int_0^\infty k^2 E(k) dk$$

$$\frac{3}{4} \left\langle (\triangle v)^2 \right\rangle = \int_0^\infty E(k) H(kr) dk \approx \int_{\pi/r}^\infty E(k) dk + \frac{r^2}{\pi^2} \int_0^{\pi/r} k^2 E(k) dk$$

$$(H(x) = 1 + 3x^{-2} \cos x - 3x^{-3} \sin x)$$

Vorticity Correlation

$$\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle = -\nabla^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$$

Spectrum Tensor

$$\Phi_{ij}(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int Q_{ij}(\boldsymbol{r})e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{r}$$

$$Q_{ij}(\boldsymbol{r}) = \int \Phi_{ij}(\boldsymbol{k})e^{j\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}$$

$$\Phi_{ij}(\boldsymbol{k}) = \frac{E(k)}{4\pi k^2}\left[\delta_{ij} - \frac{k_i k_j}{k^2}\right]$$

Signature Function

$$V(r) = -\frac{3}{8}r^2\frac{d}{dr}\frac{1}{r}\frac{d}{dr}\left\langle(\nabla v)^2\right\rangle$$

$$rV(r) \approx [kE(k)]_{k=\pi/r} \ , \qquad \int_0^\infty V(r)dr = \frac{1}{2}\left\langle \boldsymbol{u}^2\right\rangle$$

One-Dimensional Spectrum

$$E_1(k) = \frac{1}{\pi}\int_0^\infty \langle \boldsymbol{u}\cdot \boldsymbol{u}'\rangle \cos(kr)dr$$

$$E(k) = -k\frac{dE_1}{dk}$$