UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 72

FUNDAMENTALS IN FLUID MECHANICS OF CLIMATE

*You may attempt **ALL** questions, although full marks can be achieved by good answers to **THREE** questions.*
Completed answers are preferred to fragments.

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Two-dimensional small-amplitude disturbances in the $x - z$ plane, where $z > 0$ is vertically upwards, of a stationary inviscid fluid with constant buoyancy frequency N satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\frac{\partial^2 \psi}{\partial t^2} + N^2 \frac{\partial^2 \psi}{\partial x^2} = 0,$$

where $\psi(x, z, t)$ is the streamfunction.

Consider a plane wave $\psi = \hat{\psi} e^{i(kx+mz-\omega t)}, k > 0$. Find $\gamma = \gamma(k, \omega, N)$ such that $m = \gamma$ is the vertical wavenumber of a wave with positive upward group velocity.

Consider an infinite fluid with density stratification

$$N^2 = \begin{cases} N_0^2, & |z| > H/2, \\ 0, & |z| \leqslant H/2, \end{cases}$$

where $N_0^2 > 0$ is a constant.

A wave of unit amplitude is incident from $z = -\infty$ on the unstratified region. Write down the form of the solution in the regions $z < -H/2, -H/2 \leqslant z \leqslant H/2$ and $z > H/2$, respectively. Given that ψ is continuous and differentiable at $z = \pm H/2$, show that the amplitude P of the transmitted wave satisfies

$$P\left[(1 + i\gamma/k)^2 e^{-kH} - (1 - i\gamma/k)^2 e^{kH}\right] = (4i\gamma/k)e^{-i\gamma H}.$$

Hence show that the transmission coefficient T, (i.e. the fraction of the incident energy from $z < -H/2$ that crosses to $z > H/2$) is given by

$$T = \left[1 + \frac{(\gamma^2 + k^2)^2}{4k^2\gamma^2}\sinh^2 kH\right]^{-1}.$$

Show further that

$$T = \left[1 + \left(\frac{\sinh kH}{\sin 2\theta}\right)^2\right]^{-1},$$

where $\theta = \cos^{-1}(\omega/N)$, and find the angle θ corresponding to maximum energy transmission.

2

A body of water of uniform density ρ and length L with a free surface at height $z = H$ is at rest behind a dam located at $x = 0$ in a channel with a triangular cross section given by $y = \pm\frac{1}{2}\beta z$.

(a) State the assumptions embodied in the shallow-water approximation and derive the appropriate one-dimensional inviscid shallow-water equations for the triangular channel. Determine both the equations for the characteristics λ and the equations along the characteristics.

(b) Suppose that at $t = 0$ the dam begins to move with constant velocity $U_0 = \frac{4}{5}C_0$. Sketch the structure of the flow on an $x - t$ diagram and determine the time T_D and dam location X_D where the flow immediately adjacent to the dam is first affected by the finite length of the initial region of water.

(c) The speed of the dam changes from U_0 to $U_1 = U_0 + \Delta U$ at time $t = \frac{1}{2}T_D$. Describe how the structure of the flow changes for both $\Delta U > 0$ and $\Delta U < 0$. You need not provide an analytical solution for the flow, but you should describe key features and identify them in sketches. Write down any conservation relations, and state any assumptions.

3

Consider a disturbance $\eta(\boldsymbol{x}, t)$ to the free surface of a layer of inviscid fluid of constant density ρ and constant depth H, rotating with angular velocity $\frac{1}{2}f$ about the vertical $z-$axis. State the condition on η that ensures the shallow-water approximation is valid.

Write down the shallow-water equations for small disturbances and derive the conservation equation for the perturbation potential vorticity.

Suppose the fluid occupies the half-space $x > 0$ and is bounded by a vertical boundary at $x = 0$. Initially $\eta = 0$ and the velocity v in the $y-$direction is given by

$$v_0 = \begin{cases} \frac{\alpha f}{H}(L - x), & 0 \leqslant x \leqslant L, \\ 0, & x > L. \end{cases}$$

Show that in the final adjusted state η satisfies

$$\eta - a^2 \frac{\partial^2 \eta}{\partial x^2} = \begin{cases} \alpha, & 0 \leqslant x \leqslant L, \\ 0, & x > L, \end{cases}$$

where $a^2 = gH/f^2$. What is the physical significance of a?

Using conservation of mass, calculate the free surface elevation and velocity in the final state. In the limit $a \ll L$ show that the free surface at $x = 0$ intersects the bottom when $\alpha \gtrsim aH/L$, and comment on the condition this implies for v_0. Briefly discuss the flow in the limit $a \gg L$.

4

Consider a homogeneous inviscid fluid occupying $z > 0$ and moving with steady velocity $\boldsymbol{U} = (U(z), 0, 0)$ in a reference frame rotating about the vertical z axis with angular velocity $\frac{1}{2}f$.

(a) Show that the vertical velocity of a steady perturbation is governed by

$$\left[\frac{\partial^2}{\partial x^2}\nabla^2 + \frac{f^2}{U^2}\frac{\partial^2}{\partial z^2} - \frac{f^2 U'}{U^3}\frac{\partial}{\partial z} - \frac{U''}{U}\frac{\partial^2}{\partial x^2}\right] w = 0 \qquad (*)$$

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for a linear disturbance with velocity gradients in the $x - z$ plane. Hence or otherwise, show that solutions to (*) are purely oscillatory only if $U' = dU/dz = 0$.

(b) Outline the WKB approximation as it applies to wave solutions in this flow. Under the assumption that U' and U'' can be neglected locally in (*), determine the functional relationship between m and k, where $\boldsymbol{k} = (k, 0, m)$ is the wavenumber vector for an upward propagating wave. State any restrictions for the wavenumber components if $U(0) > 0$. Give a relationship between the wavenumber vector $\boldsymbol{k}$ and the phase velocity $\boldsymbol{c}_p$ (in the stationary frame) that describes their relative orientations. Hence or otherwise, derive a differential equation for $X(z)$ such that the phase of the wave is constant along $x = X(z)$. Sketch contours of constant phase (i) if $U(z)$ increases with height and (ii) if $U(z)$ decreases with height. In each case, determine if a limiting height is reached and describe the behaviour at that height.

(c) Suppose the velocity profile is given by

$$U(z) = \begin{cases} U_0, & 0 \leqslant z \leqslant H, \\ U_1, & z > H. \end{cases}$$

What is conserved at $z = H$? Sketch contours of constant phase for the cases of $U_1 > U_0$ and $U_1 < U_0$. Discuss any limits on U_1.

END OF PAPER