MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012  9:00 am to 12:00 pm

PAPER 70

APPROXIMATION THEORY

Attempt no more than **FOUR** questions.
There are **SIX** questions in total.
The questions carry equal weight.

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**STATIONERY REQUIREMENTS**
- Cover sheet
- Treasury Tag
- Script paper

**SPECIAL REQUIREMENTS**
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
Let $j_n$ be the Jackson operator, i.e., for a $2\pi$-periodic function $f$ from $C(\mathbb{T})$,

$$j_n(f, x) := \int_{-\pi}^{\pi} f(x-t)J_n(t) \, dt, \quad J_n(t) := \frac{3}{2\pi n (2n^2 + 1)} \sin^4 \frac{nt}{2}, \quad \int_{-\pi}^{\pi} J_n(t) \, dt = 1.$$ 

Prove that, for any $f \in C(\mathbb{T})$, we have the estimate

$$\|j_n(f) - f\| \leq c \omega_2(f, \frac{1}{n}),$$

where $\omega_2(f, t)$ is the second modulus of smoothness of $f$.

Hence, derive that if $f$ is twice continuously differentiable, then

$$E_n(f) \leq \frac{c_1}{n^2} \|f''\|_{C(\mathbb{T})}.$$ 

1

a) For $f \in C[0, 1]$, write down the definition of the Bernstein polynomial $B_n(f)$ of degree $n$, and prove that $\|B_n(f)\|_{\infty} \leq \|f\|_{\infty}$.

b) For a function $f \in C[0, 1]$ that takes integer values at $x = 0$ and $x = 1$, and for the sequence of polynomials

$$B^*_n(f, x) := \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n} \right) x^k (1-x)^{n-k},$$

prove that $\|B_n(f) - B^*_n(f)\|_{\infty} \to 0$ as $n \to \infty$. Here, $[t]$ is the largest integer not bigger than $t$.

c) Hence show that a function $f \in C[0, 1]$ is approximable by polynomials with integral coefficients if and only if $f(0)$ and $f(1)$ are integers.
3

a) State the Chebyshev alternation theorem for the element of best uniform approximation to a $2\pi$-periodic function $f \in C(T)$ from $T_n$, the space of all trigonometric polynomials of degree $n$.

b) Let

$$f(x) = \sum_{k=0}^{\infty} a_k \cos 5^k x, \quad a_k > 0, \quad \sum_{k=0}^{\infty} a_k < \infty.$$  

Prove that, for $5^m \leq n < 5^{m+1}$, the polynomial

$$t_n(x) = \sum_{k=0}^{m} a_k \cos 5^k x$$

is the best approximant to $f$ from $T_n$ and find the value of $E_n(f)$ in terms of $a_k$.

c) Let $a_k = \frac{1}{3^k}$. Find the value of $\alpha$ such that

$$E_n(f) \leq \frac{C}{n^\alpha} \quad \forall n \in \mathbb{N}.$$  

4

For a knot sequence $(t_i)_{i=1}^{n+k} \subset [a,b]$ with distinct knots, let

$$M_i(t) := k[t_i, \ldots, t_{i+k}](t - t)^{k-1}, \quad N_i(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](t - t)^{k-1}$$

be the sequences of $L_1$- and $L_\infty$-normalized B-splines, respectively.

a) Prove that $M_i$ are the piecewise-polynomial functions of degree $k-1$ and global smoothness $C^{k-2}$, with knots $(t_i, \ldots, t_{i+k})$ and with the finite support $[t_i, t_{i+k}]$.

b) Prove that

$$\int_{t_i}^{t_{i+k}} M_i(t) \, dt = 1,$$

and that

$$\sum_{i=1}^{n} N_i(t) \equiv 1, \quad t_k < t < t_{n+1}.$$  

Part III, Paper 70
Let \((N_i)\) and \((M_i)\) be the B-splines bases of degree \(k - 1\) with \(L_{\infty}\) and \(L_1\)-normalization, respectively, defined on a knot sequence \(\Delta = (t_i)_{i=1}^{n+k} \subset [0, 1]\).

Given \(f \in C[0, 1]\), let
\[
P_S(f) := s^* = \sum_{j=1}^{n} a_j N_j
\]
be the orthogonal projection of \(f\) onto \(S := \text{span}(N_j)\) with respect to the ordinary inner product \((f, g) = \int_0^1 f(x)g(x)\,dx\). Then \(P_S\) is also well defined as an operator from \(C[0, 1]\) onto \(C[0, 1]\).

a) Show that the max-norm of \(P_S\) satisfies the inequality
\[
\|P_S\|_\infty \leq \|G^{-1}\|_{\ell_\infty},
\]
where \(G = (g_{ij})\) is the Gram matrix with the elements \(g_{ij} = (M_i, N_j)\).

b) For linear splines \((k = 2)\) and equidistant \(\Delta\), with \(t_{i+1} - t_i = h\) for all \(i\), compute the entries of \(G\).

c) Using the fact that \(G\) is totally positive, or otherwise, prove the estimate
\[
\|G^{-1}\|_{\ell_\infty} \leq 3, \quad k = 2.
\]
[You may use any appropriate theorems on the inverse of certain matrices if correctly stated.]
a) Define a multiresolution analysis of $L_2(\mathbb{R})$ with a generator $\phi$ and explain how it is related to existence of an orthonormal wavelet $\psi$.

b) Prove that the following properties of $\phi$

1) $\phi(x) = \sum_n a_n \phi(2x - n)$,  
2) $\{\phi(\cdot - n)\}_{n \in \mathbb{Z}}$ is an orthonormal sequence

are equivalent to

1') $f(2t) = m(t)f(t), \quad m(t) = \frac{1}{2} \sum_n a_n e^{-int},$

2') $\sum |f(t + 2\pi k)|^2 \equiv 1 \text{ a.e.}$

where $f$ is the Fourier transform of $\phi$, i.e., $f(t) = \hat{\phi}(t) = \int_{\mathbb{R}} \phi(x)e^{-ixt} \, dx$.

c) Verify that conditions 1') - 2') are fulfilled for the function $f = \hat{\phi}$ defined as

$$f = \begin{cases} 1, & t \in [-\pi, \pi) \\ 0, & \text{otherwise} \end{cases}$$

Using the inverse Fourier transform or otherwise, determine the corresponding generator $\phi$.

END OF PAPER