UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 9:00 am to 12:00 pm

PAPER 70

APPROXIMATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let j_n be the Jackson operator, i.e., for a 2π-periodic function f from $C(\mathbb{T})$,

$$j_n(f,x) := \int_{-\pi}^{\pi} f(x-t) J_n(t)\,dt, \quad J_n(t) := \frac{3}{2\pi n(2n^2+1)} \frac{\sin^4 \frac{nt}{2}}{\sin^4 \frac{t}{2}}, \quad \int_{-\pi}^{\pi} J_n(t)\,dt = 1.$$

Prove that, for any $f \in C(\mathbb{T})$, we have the estimate

$$\|j_n(f) - f\| \leqslant c\,\omega_2(f, \tfrac{1}{n}),$$

where $\omega_2(f,t)$ is the second modulus of smoothness of f.

Hence, derive that if f is twice continuously differentiable, then

$$E_n(f) \leqslant \frac{c_1}{n^2} \|f''\|_{C(\mathbb{T})}.$$

2

a) For $f \in C[0,1]$, write down the definition of the Bernstein polynomial $B_n(f)$ of degree n, and prove that $\|B_n(f)\|_\infty \leqslant \|f\|_\infty$.

b) For a function $f \in C[0,1]$ that takes integer values at $x=0$ and $x=1$, and for the sequence of polynomials

$$B_n^*(f,x) := \sum_{k=0}^{n} \left\lfloor \binom{n}{k} f\left(\frac{k}{n}\right) \right\rfloor x^k (1-x)^{n-k},$$

prove that $\|B_n(f) - B_n^*(f)\|_\infty \to 0$ as $n \to \infty$. Here, $\lfloor t \rfloor$ is the largest integer not bigger than t.

c) Hence show that a function $f \in C[0,1]$ is approximable by polynomials with integral coefficients if and only if $f(0)$ and $f(1)$ are integers.

3

a) State the Chebyshev alternation theorem for the element of best uniform approximation to a 2π-periodic function $f \in C(\mathbb{T})$ from $\mathcal{T}_n$, the space of all trigonometric polynomials of degree n.

b) Let

$$f(x) = \sum_{k=0}^{\infty} a_k \cos 5^k x, \qquad a_k > 0, \qquad \sum_{k=0}^{\infty} a_k < \infty.$$

Prove that, for $5^m \leqslant n < 5^{m+1}$, the polynomial

$$t_n(x) = \sum_{k=0}^{m} a_k \cos 5^k x$$

is the best approximant to f from $\mathcal{T}_n$ and find the value of $E_n(f)$ in terms of a_k.

c) Let $a_k = \frac{1}{3^k}$. Find the value of α such that

$$E_n(f) \leqslant \frac{c}{n^\alpha} \quad \forall n \in \mathbb{N}\,.$$

4

For a knot sequence $(t_i)_{i=1}^{n+k} \subset [a,b]$ with distinct knots, let

$$M_i(t) := k[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}, \qquad N_i(t) := (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}](\cdot - t)_+^{k-1}$$

be the sequences of L_1- and L_∞-normalized B-splines, respectively.

a) Prove that M_i are the piecewise-polynomial functions of degree $k-1$ and global smoothness C^{k-2}, with knots $(t_i, \ldots, t_{i+k})$ and with the finite support $[t_i, t_{i+k}]$.

b) Prove that

$$\int_{t_i}^{t_{i+k}} M_i(t)\, dt = 1\,,$$

and that

$$\sum_{i=1}^{n} N_i(t) \equiv 1, \qquad t_k < t < t_{n+1}\,.$$

5

Let (N_i) and (M_i) be the B-splines bases of degree $k-1$ with L_∞- and L_1-normalization, respectively, defined on a knot sequence $\Delta = (t_i)_{i=1}^{n+k} \subset [0,1]$.

Given $f \in C[0,1]$, let

$$P_{\mathcal{S}}(f) := s^* = \sum_{j=1}^{n} a_j N_j$$

be the orthogonal projection of f onto $\mathcal{S} := \operatorname{span}(N_j)$ with respect to the ordinary inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$. Then $P_{\mathcal{S}}$ is also well defined as an operator from $C[0,1]$ onto $C[0,1]$.

a) Show that the max-norm of $P_{\mathcal{S}}$ satisfies the inequality

$$\|P_{\mathcal{S}}\|_\infty \leqslant \|G^{-1}\|_{\ell_\infty},$$

where $G = (g_{ij})$ is the Gram matrix with the elements $g_{ij} = (M_i, N_j)$.

b) For linear splines ($k=2$) and equidistant Δ, with $t_{i+1} - t_i = h$ for all i, compute the entries of G.

c) Using the fact that G is totally positive, or otherwise, prove the estimate

$$\|G^{-1}\|_{\ell_\infty} \leqslant 3, \qquad k = 2.$$

[You may use any appropriate theorems on the inverse of certain matrices if correctly stated.]

6

a) Define a multiresolution analysis of $L_2(\mathbb{R})$ with a generator ϕ and explain how it is related to existence of an orthonornal wavelet ψ.

b) Prove that the following properties of ϕ

$$1) \quad \phi(x) = \sum_n a_n \phi(2x - n), \qquad 2) \quad \{\phi(\cdot - n)\}_{n \in \mathbb{Z}} \quad \text{is an orthonormal sequence}$$

are equivalent to

$$1') \qquad f(2t) = m(t) f(t)\,, \qquad m(t) = \tfrac{1}{2} \sum_n a_n e^{-int},$$

$$2') \qquad \sum |f(t + 2\pi k)|^2 \equiv 1 \text{ a.e.}$$

where f is the Fourier transform of ϕ, i.e., $f(t) = \widehat{\phi}(t) = \int_{\mathbb{R}} \phi(x) e^{-ixt}\, dx$.

c) Verify that conditions $1') - 2')$ are fulfilled for the function $f = \widehat{\phi}$ defined as

$$f = \begin{cases} 1, & t \in [-\pi, \pi) \\ 0, & \text{otherwise}\,. \end{cases}$$

Using the inverse Fourier transform or otherwise, determine the corresponding generator ϕ.

END OF PAPER