UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 6 June, 2012 1:30 pm to 4:30 pm

PAPER 7

MATHEMATICAL TOPICS IN KINETIC THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

We shall consider in this exercise the dispersive and averaged estimates on a transport equation. Let us consider the equation

$$\frac{\partial f}{\partial t} + a(v) \cdot \nabla_x f = 0 \tag{1}$$

on the function $f = f(t,x,v) \geqslant 0$ with $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, $d \in \mathbb{N}^*$, and where $a : \mathbb{R}^d \to \mathbb{R}^d$ is a C^∞ function such that its Jacobian j_a (determinant of its Jacobian matrix J_a) satisfies

$$\forall\, v \in \mathbb{R}^d, \quad 0 < \alpha_1 \leqslant |j_a(v)| \leqslant \alpha_2 \tag{2}$$

for two positive constants $\alpha_1, \alpha_2 > 0$.

(a) Define the characteristics ODE system and the characteristics map associated with the equation, and solve these equations in order to obtain an explicit formula for the characteristics map.

(b) Deduce that for any initial data $f_{in} \in C_c^\infty$, the equation (1) has global solutions defined by

$$f(t,x,v) = f_{in}(x - ta(v), v) \tag{3}$$

and that this global solution is C^∞ in t, x, v with compact support in x, v for any time $t \geqslant 0$.

(c) Prove also that for any $T > 0$ there is $R > 0$ such that the support of the solution $f(t,\cdot,\cdot)$ constructed in (3) is included in the ball $B(0,R) \subset \mathbb{R}^d \times \mathbb{R}^d$ for any $0 \leqslant t \leqslant T$. Let us denote

$$\mathcal{C}_T = \Big\{ f(t,x,v) \in C^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^d) \;; \\ \exists\, R_T > 0 \mid \forall\, t \in [0,T],\ \text{support} f(t,\cdot,\cdot) \subset B(0,R_T) \Big\}. \tag{4}$$

(d) Prove that for any initial data $f_{in} \in C_c^\infty$ the solution to the equation (1) on $[0,T]$ is unique within the class $\mathcal{C}_T$.

[*Hint: Consider the L^2 norm in x and v of the difference of two solutions, prove that it is well-defined and time-differentiable, and compute its time derivative.*]

(e) Recall the statement of the dispersion estimate for the free transport equation, i.e. in the case when $a(v) = v$.

(f) Prove the following inequality

$$\|f(t,\cdot,\cdot)\|_{L^\infty_x L^1_v} = \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |f(t,x,v)|\, dv \leqslant \frac{C_a}{t^d} \|f(t,\cdot,\cdot)\|_{L^1_x L^\infty_v} \\ = \frac{C_a}{t^d} \int_{\mathbb{R}^d} \left(\sup_{v \in \mathbb{R}^d} |f(t,x,v)| \right) dx \tag{5}$$

and give an expression of the constant $C_a > 0$ in terms of the assumptions made on the Jacobian of a.

(g) Consider a solution to the equation (1) on $[0, T]$ within the class $\mathcal{C}_T$ described above. Prove that the function $g(t, x, v) = tJ_a(v)\nabla_x f(t, x, v) + \nabla_v f(t, x, v)$ is also solution to the equation (1) on $[0, T]$ within the class $\mathcal{C}_T$, where $J_a(v)$ is the Jacobian matrix of $a(v)$ at point v. Deduce that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |TJ_a(v)\nabla_x f(T, x, v) + \nabla_v f(T, x, v)|^2 \,\mathrm{dx}\,\mathrm{dv} = \int_{\mathbb{R}^\mathrm{d} \times \mathbb{R}^\mathrm{d}} |\nabla_\mathrm{v} \mathrm{f}_\mathrm{in}(\mathrm{x}, \mathrm{v})|^2 \,\mathrm{dx}\,\mathrm{dv}.$$

(6)

2

This question is concerned with the two-dimensional incompressible Euler equation in vortices formulation

$$\partial_t \omega + u \cdot \nabla_x \omega = 0, \quad \omega = \omega(t,x) \in \mathbb{R}, \; x \in \mathbb{R}^2, \tag{1}$$

where u is defined in terms of ω by

$$u = \nabla_x^{\perp} \phi = (-\partial_{x_2}\phi, \partial_{x_1}\phi) \quad \text{and} \quad \phi(t,x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y|\, \omega(t,y)\, \mathrm{d}y,$$

and with some initial conditions $\omega_{in} \in C_c^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$. This is a nonlinear transport equation sharing structural similarity with the Vlasov-Poisson equation. We are interested here in a theorem due to Yudovich about the existence and uniqueness of solution to this equation. We shall only be concerned with the core part of the proof of this theorem, i.e. the construction of the characteristics, and we shall assume the existence of a global solution $\omega(t,x)$ which is $L^1 \cap L^\infty(\mathbb{R}^2)$ for any $t \geqslant 0$.

(a) Assume that ω is moreover C^2 with compact support, and then prove that the so-called *stream function* ϕ defined above is C^2 and satisfies the Poisson equation $\Delta_x \phi = \omega$.

(b) Write the characteristics system of this transport equation in Hamiltonian form, and give the corresponding Hamiltonian.

(c) Assuming the existence of a global C^2 solution to the characteristics system, define the characteristics map $S_{0,t}$ and prove that its Jacobian is equal to one for any $t \geqslant 0$.

(d) We define for any function $\Omega \in L^1 \cap L^\infty(\mathbb{R}^2)$ the new function $U[\Omega]$ on $\mathbb{R}^2$ by

$$U[\Omega](x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \Omega(y)\, \mathrm{d}y.$$

Check this formula yields a well-defined function $U[\Omega] \in L^\infty(\mathbb{R}^2)$.

(e) Prove the following functional inequality

$$\begin{aligned}\forall\, x_1, x_2 \in \mathbb{R}^2, \quad & |U[\Omega](x_1) - U[\Omega](x_2)| \\ & \leqslant C \left(\|\Omega\|_{L^1(\mathbb{R}^2)} + \|\Omega\|_{L^\infty(\mathbb{R}^2)} \right) |x_1 - x_2| \, \ln|x_1 - x_2|\end{aligned}$$

for some constant $C > 0$.

[*Hint: Assume w.l.o.g. that* $|x_1 - x_2| \leqslant e^{-1}$, *and split the integral in three parts according to the region* $|y - x_3| \geqslant 1$ *and* $M \leqslant |y - x_3| \leqslant 1$ *and* $|y - x_3| \leqslant M$ *with* $x_3 = (x_1 + x_2)/2$ *and* $M = |x_1 - x_2| \ln|x_1 - x_2|$. *Show that the first part is controlled by a constant times* $|x_1 - x_2| \|\Omega\|_{L^1}$, *the second part by a constant times* $|x_1 - x_2| \ln M \|\Omega\|_{L^\infty}$, *and the third part by a constant times* $M \|\Omega\|_{L^\infty}$.]

(f) Assuming that a given force field $U : \mathbb{R}^2 \to \mathbb{R}^2$ is $C^0 \cap L^\infty$ and satisfies the so-called "Log-Lipschitz" condition

$$\forall\, x_1, x_2 \in \mathbb{R}^2, \quad |U(x_1) - U(x_2)| \leqslant C\, |x_1 - x_2| \, \ln|x_1 - x_2|$$

prove that the ODE system

$$\dot{x}(t) = U(x(t)), \quad x(t) = (x_1(t), x_2(t))$$

has global unique solution for any initial initial data (y_1, y_2).

[*Hint: We shall admit the existence of local C^1 solution which follows from the Cauchy-Peano theorem. In order to prove uniqueness, establish a Gronwall estimate on the difference of two solutions.*]

(g) Conclude that the characteristics are globally uniquely defined for any solution to (1) with $\omega \in L^1 \cap L^\infty(\mathbb{R}^2)$.

3

This exercise is concerned with the Weyl theorem. We consider a Hilbert space H and a bounded operator L on H and a *compact operator* K on H, i.e. K is bounded and maps the unit ball into a precompact set. We recall that a bounded operator L on H is called *Fredholm* if

1. $\ker(L)$ is finite-dimensional;
2. $\operatorname{coker}(L) = H/\operatorname{range}(L)$ is finite-dimensional;
3. $\operatorname{range}(L)$ is closed.

A complex number ξ is said to in the *essential spectrum* of L iff $(L - \xi)$ is not Fredholm.

(a) Prove Riesz's theorem: a Hilbert space is finite-dimensional if and only if its closed unit ball is strongly compact.

[*Hint: For proving the infinite-dimensional version of the statement, construct a sequence h_n such that $\|h_n - h_m\| \geqslant 1/2$ as soon as m differs from n, which contradicts the compactness property.*]

(b) Define what it means for a bounded operator to be *self-adjoint*. Define the *spectrum* of a bounded operator.

(c) Prove that for a bounded operator L on H, the spectrum is bounded, and when L is self-adjoint, $\Sigma(L) \subset \mathbb{R}$.

(d) Prove that a bounded operator K on H is compact iff it has the following complete continuity property: for any h_n weakly convergent in H, then Kh_n is strongly convergent.

(e) Prove that the spectrum of a compact operator K on H is either finite or it is a countable sequence converging to zero. Moreover prove that any non-zero eigenvalue is associated with a finite-dimensional eigenspace.

(f) Consider a bounded operator L and a compact operator K on H. Prove that L is Fredholm iff $L + K$ is Fredholm.

(g) Consider a bounded operator L and a compact operator K on H. Prove that L and $L + K$ have the same essential spectrum.

4

Write an essay on the nonlinear Boltzmann equation, including a presentation and as many properties, remarks, statements and proofs as possible.

END OF PAPER