UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 69

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt no more than ***THREE*** *questions from Section I*
and ***ONE*** *from Section II.*

There are ***SEVEN*** *questions in total.*

The questions in Section II carry twice the weight of those in Section I.
Questions within each Section carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

The equation

$$(py'')'' - (qy')' + ry = f$$

is given in the interval $[-1,1]$ with the zero boundary conditions $y(\pm 1) = y'(\pm 1) = 0$. Here $p, q, r \in C[-1,1]$, $f \in L_2[-1,1]$, $p(x) > 0$, $q(x), r(x) \geqslant 0$ for all $x \in [-1,1]$.

1. Rewrite this as a variational problem, determining the corresponding variational functional and identifying the correct function space $\mathcal{H}$ where it should be minimised.

2. Prove that this variational problem possesses a single minimum in $\mathcal{H}$, which is the weak solution of the underlying ODE system.

Fully justify each step by quoting precise definitions and statements of relevant theorems.

2

We are concerned with the three-stage Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{5}{24} & \frac{1}{3} & -\frac{1}{24} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}.$$

1. Determine the order of this method.

2. Is the method A-stable?

3. Is it algebraically stable?

3

Consider the initial-value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x}, \qquad 0 \leqslant x \leqslant 1, \quad t \geqslant 0\,,$$

given with initial conditions on $[0,1]$ and periodic boundary conditions at 0 and 1. α is a given real constant.

1. Determine the order of approximation of the semi-discretized scheme

$$u_m' = \frac{1}{(\Delta x)^2}(u_{m-1} - 2u_m + u_{m+1}) + \frac{\alpha}{2\Delta x}(u_{m+1} - u_{m-1}), \qquad m = 1, \ldots, N\,,$$

   where $\Delta x = 1/N$.

2. Prove stability using two techniques out of {eigenvalue analysis, Fourier analysis, the energy method}.

4

The ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is solved with the two-step method

$$\mathbf{y}_{n+2} - \alpha \mathbf{y}_{n+1} - (1-\alpha)\mathbf{y}_n = h[(1 - \tfrac{1}{4}\alpha)\mathbf{f}(\mathbf{y}_{n+2}) + (1 - \tfrac{3}{4}\alpha)\mathbf{f}(\mathbf{y}_n)]\,.$$

1. What is the order of the method for different values of α? For which values of α is the method convergent?

2. Determine the values of α for which the method is convergent and A-stable.

5

The diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2$, $x \in [0,1]$, $t \geqslant 0$, given with L_2 initial conditions and zero Dirichlet boundary conditions, is solved by the numerical method

$$\begin{aligned} & -a\mu u_{m-1}^{n+1} + (1+2a\mu)u_m^{n+1} - a\mu u_{m+1}^{n+1} \\ = \; & (1-a)\mu u_{m-1}^n + [1-2(1-a)\mu]u_m^n + (1-a)\mu u_{m+1}^n \,, \end{aligned}$$

where $m = 1, \ldots, M$, $n \geqslant 0$, $\Delta x = 1/M$, $\mu = \Delta t/(\Delta x)^2$ and a is a given real constant.

1. Determine the order of the method for every a;

2. Determine the range of a for which the method is stable for every $\mu > 0$.

SECTION II

6

Write an essay on error control methodologies for discretised ordinary differential equations.

7

Write an essay on stability analysis of numerical methods for partial differential equations of evolution using Fourier analysis.

END OF PAPER