UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 9:00 am to 11:00 am

PAPER 68

DISTRIBUTION THEORY AND APPLICATIONS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let P be an Nth order polynomial in $\lambda = (\lambda_1, \ldots, \lambda_n)$. What does it mean to say that $P(D)$ is elliptic? Show that if $P(D)$ is elliptic then there exists a $C > 0$ such that $|P(\lambda)| \geqslant C\langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large.

Define the Sobolev space $H^s(\mathbf{R}^n)$ and the local Sobolev space $H^s_{\rm loc}(X)$, where $X \subset \mathbf{R}^n$ is open. Prove that if $u \in \mathcal{D}'(\mathbf{R}^n)$ and u has compact support then $u \in H^t(\mathbf{R}^n)$ for some $t \in \mathbf{R}$.

Suppose $P(D)$ is elliptic. If $u \in \mathcal{D}'(X)$ and $P(D)u \in H^s_{\rm loc}(X)$, prove that $u \in H^{s+N}_{\rm loc}(X)$.

Now let Q be an Nth order polynomial in $\lambda = (\lambda_1, \ldots, \lambda_n)$. Suppose that there exist constants $C, \delta > 0$ such that $|\partial^\alpha Q(\lambda)| \leqslant C|\lambda|^{-\delta|\alpha|}|Q(\lambda)|$ for all multi-indices α and for $|\lambda|$ sufficiently large. Explain how you would adjust your proof in the previous part to show that

$$Q(D)u \in H^s_{\rm loc}(X) \Rightarrow u \in H^{s+\delta N}_{\rm loc}(X).$$

[You may use elementary facts about Sobolev spaces provided they are clearly stated.]

2

Let $X \subset \mathbf{R}^n$ be open. Define the class of symbols $\mathrm{Sym}(X, \mathbf{R}^k; N)$. What properties must a function $\Phi : X \times \mathbf{R}^k \to \mathbf{R}$ satisfy for it to be a phase function?

If $a \in \mathrm{Sym}(X, \mathbf{R}^k; N)$ and Φ is a phase function, define what is meant by the oscillatory integral

$$I_\Phi(a) = \int e^{\mathrm{i}\Phi(x,\theta)} a(x, \theta)\, \mathrm{d}\theta.$$

You may assume that $I_\Phi(a) \in \mathcal{D}'(X)$. Define the singular support of a distribution in $\mathcal{D}'(X)$. Show that

$$\operatorname{sing\,supp} I_\Phi(a) \subset \Big\{x : \nabla_\theta \Phi(x, \theta) = 0 \ \text{ for some } \theta \in \mathbf{R}^k \cap \operatorname{supp} a(x, \cdot)\Big\}.$$

Consider the initial value problem

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} - \Delta_x E = 0 \qquad (x, t) \in \mathbf{R}^{n+1}$$

$$E = 0, \quad \frac{\partial E}{\partial t} = \delta_0(x) \qquad \text{when } t = 0.$$

Show that the solution can be written as the sum of an ordinary function and two oscillatory integrals with symbols in $\mathrm{Sym}(\mathbf{R}^{n+1}, \mathbf{R}^n; -1)$. Show that $\operatorname{sing\,supp} E$ is contained in the light cone $\{(x, t) : |x| = c|t|\}$.

3

Define the space of Schwartz functions $\mathcal{S}(\mathbf{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$. Show that the Fourier transform

$$\mathcal{F} : \varphi \mapsto \hat{\varphi}(\lambda) = \int e^{-\mathrm{i}\lambda\cdot x}\varphi(x)\,\mathrm{d}x$$

defines a continuous isomorphism on $\mathcal{S}(\mathbf{R}^n)$. Define the Fourier transform on $\mathcal{S}'(\mathbf{R}^n)$ and show that this defines a continuous isomorphism on $\mathcal{S}'(\mathbf{R}^n)$.

Let G be a real, symmetric, positive definite 3×3 matrix with entries g^{ij}. Consider the quadratic form $g : \mathbf{R}^3 \to \mathbf{R}$ defined by

$$g(x) = \sum_{i,j=1}^{3} g^{ij} x_i x_j .$$

Show that the linear form

$$\varphi \mapsto \langle (1/g), \varphi\rangle = \int \frac{\varphi(x)}{g(x)}\,\mathrm{d}x, \qquad \forall \varphi \in \mathcal{S}(\mathbf{R}^3)$$

defines a tempered distribution and compute its Fourier transform $(1/g)\hat{}(\lambda)$. Briefly explain how you would extend this result to quadratic forms associated with matrices whose real part is G. Hence show that

$$\left[\frac{1}{x_1^2 + x_2^2 + 2x_3^2 + 2\mathrm{i}x_1x_2}\right]\hat{}(\lambda) = \frac{\sqrt{2}\pi^2}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\mathrm{i}\lambda_1\lambda_2}},$$

where the square-root is defined to have positive real part.

[You may use the fact that $\int_0^\infty x^{-1}e^{-ax}\sin x\,\mathrm{d}x = \arctan(1/a)$ for $a > 0$. Elementary results from linear algebra may be used without proof.]

END OF PAPER