UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 12 June, 2012 9:00 am to 11:00 am

PAPER 66

QUANTUM FOUNDATIONS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Alice and Bob (located at $\mathbf{x_A}$ and $\mathbf{x_B}$ respectively) share a maximally entangled state $|\Phi^+\rangle_{d_1 d_2} = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle_{d_1}|\uparrow_z\rangle_{d_2} + |\downarrow_z\rangle_{d_1}|\downarrow_z\rangle_{d_2})$ of two spin-$\frac{1}{2}$ particles d_1 and d_2. In addition they hold spin-$\frac{1}{2}$ particles A and B prepared in advance by a third party in an unknown state $|\psi\rangle_{AB}$. (In the following it is assumed that Alice and Bob each complete their local operations and measurements during time $\Delta t \ll L/c$, where $L = |\mathbf{x_A} - \mathbf{x_B}|$.)

(a) Describe an explicit protocol which will allow Alice and Bob to perform an instantaneous non-demolition verification that $|\psi\rangle_{AB}$ is a state of zero total spin.

(b) Write down the product eigenstates and eigenvalues of the operator

$$(\sigma_z^A \otimes \mathbb{I}^B + \mathbb{I}^A \otimes \sigma_z^B) mod\ 4 \tag{1}$$

and use the result obtained in (a) to show how to perform an instantaneous non-demolition measurement of (1).

(c) Hence suggest a procedure to perform an instantaneous non-demolition measurement of the Bell-operator, i.e. show how the four Bell-states $|\Phi^\pm\rangle_{AB}$, $|\Psi^\pm\rangle_{AB}$ can be distinguished with certainty without being disturbed. [You may assume that Alice and Bob share two maximally entangled states of the type $|\Phi^+\rangle_{d_1 d_2}$ as a resource.]

(d) Consider an operator on the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ with the following eigenstates

$$\begin{aligned}
|\psi_1\rangle_{AB} &= |\uparrow_z\rangle_A \otimes |\uparrow_z\rangle_B \\
|\psi_2\rangle_{AB} &= |\uparrow_z\rangle_A \otimes |\downarrow_z\rangle_B \\
|\psi_3\rangle_{AB} &= |\downarrow_z\rangle_A \otimes (\cos\theta|\uparrow_z\rangle_B + \sin\theta|\downarrow_z\rangle_B) \\
|\psi_4\rangle_{AB} &= |\downarrow_z\rangle_A \otimes (\sin\theta|\uparrow_z\rangle_B - \cos\theta|\downarrow_z\rangle_B),
\end{aligned} \tag{2}$$

where $0 \leqslant \theta \leqslant \pi/4$.
Show that the possibility of an instantaneous non-demolition measurement of (2) would contradict relativistic causality unless $\theta = 0$. Is there a simple way to perform such a measurement in the latter case? Comment on your result.

2

(a) Explain briefly what is meant by *quantum teleportation* of an unknown qubit, stating clearly what quantum and classical resources are used.

(b) Alice holds a qubit A, which is in an unknown pure entangled state with some external quantum system. Give an explicit description of the quantum teleportation protocol of the qubit A from Alice to Bob. Explain what happens with the original correlation of A as a result.

(c) Let $|0\rangle, |1\rangle, \ldots, |n-1\rangle$ be an orthonormal basis of an n-dimensional Hilbert space, let ω be a complex n-th root of unity, and define $|r\rangle = |r \ (\bmod n)\ \rangle$ for $r \geqslant n$. Show that the states

$$|\psi_{rs}\rangle_{AB} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega^{jr} |j\rangle_A |j+s\rangle_B \tag{1}$$

of two n-level quantum systems are orthonormal. Using these states, develop a protocol for teleporting an unknown state in an n-dimensional Hilbert space. [You may assume that the unknown state is pure. You may find the summation formulae for ω

$$\sum_{k=0}^{n-1} \omega^k = \begin{cases} 1, & \text{for } n = 1, \\ 0, & \text{for } n > 1, \end{cases} \tag{2}$$

useful.]

(d) Alice and Bob share a single entangled state $|\psi_{00}\rangle_{AB_1}$. Bob and Clare share a single entangled state $|\psi_{00}\rangle_{B_2C}$. Assume that Alice, Bob and Clare can perform any local operations they wish in their laboratories and are allowed to use classical communication. They do not share any additional entangled states. Develop a protocol, which will allow them to create a state $|\psi_{00}\rangle_{AC}$ shared between Alice and Clare. You may use the result obtained in (c).

3

(a) State EPR's criterion for identifying an element of physical reality.

(b) Consider an experiment in which two space-like separated parties, Alice and Bob, perform measurements on their local 2-level systems, which interacted in the past. We assume that each party may choose between two measurement settings a, a' and b, b' respectively. Each measurement yields an outcome ± 1. The experiment is characterised by the expectation values $\mathbb{E}(a,b)$, $\mathbb{E}(a,b')$, $\mathbb{E}(a',b)$, and $\mathbb{E}(a',b')$ of the product of the outcomes of local measurements. Explain how the criterion described in (a) leads to the assumption of a local hidden variables (LHV) model for such an experiment and explain its meaning. Under this assumption derive the CHSH-Bell inequality

$$\mid \mathbb{E}(a,b) - \mathbb{E}(a,b') \mid + \mid \mathbb{E}(a',b) + \mathbb{E}(a',b') \mid \leqslant 2. \tag{1}$$

(c) Now assume that the systems held by Alice and Bob are spin-$\frac{1}{2}$ particles in the joint state

$$\rho(x)_{AB} = \left(\frac{1}{2} + x\right) |\Phi^+\rangle\langle\Phi^+| + \left(\frac{1}{2} - x\right) |\Phi^-\rangle\langle\Phi^-|, \tag{2}$$

where $|\Phi^{\pm}\rangle$ are corresponding Bell-states and $0 \leqslant x \leqslant \frac{1}{2}$. Show that the correlations in $\rho(0)_{AB}$ can be simulated by a trivial LHV model. Hence, without performing calculations, deduce that $\rho(0)_{AB}$ will not violate CHSH inequality. Explain.

(d) Find the range of values of x, for which the correlations in $\rho(x)_{AB}$ do not admit LHV description. Comment on your result. [You may use the formula for the expectation value of an operator $\hat{A}$ for a mixed state ρ: $\langle \hat{A} \rangle = Tr(\hat{A}\rho)$.]

4

In the standard von Neumann model of a projective measurement of a 2-level quantum system the initial product state of the system and the measurement device $|\Psi_{in}\rangle_{SD} = \left(\sum_{i=0}^{1} c_i |i\rangle_S\right) \otimes |0\rangle_D$ is transformed into $|\Psi_f\rangle_{SD} = \sum_{i=0}^{1} c_i |i\rangle_S |i\rangle_D$, where $\langle\, i \,|\, j \,\rangle = \delta_{ij}$. This transformation is governed by the Hamiltonian of interaction

$$H_{int} = g(t)\frac{\pi}{4}\left(\mathbb{I}^S \otimes \mathbb{I}^D - \sigma_z^S \otimes \mathbb{I}^D\right)\left(\mathbb{I}^S \otimes \mathbb{I}^D - \mathbb{I}^S \otimes \sigma_x^D\right), \tag{1}$$

where $g(t)$ is a time-dependent coupling coefficient, which is non-zero only between times t_1 and t_2, such that

$$\int_{t_1}^{t_2} g(t)dt = 1.$$

Here $|0\rangle$, $|1\rangle$ are eigenstates of σ_z in the corresponding Hilbert spaces.

(a) Show, via explicit calculation, that the action of H_{int} results in a CNOT-transformation

$$U_{SD}^{CNOT} = |0\rangle\langle 0|_S \otimes \mathbb{I}^D + |1\rangle\langle 1|_S \otimes \sigma_x^D \tag{2}$$

being implemented on S and D, with S as a control and D as a target (assume $\hbar = 1$).

(b) By changing the basis to the complementary basis $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ for both S and D show that (2) is equivalent to the CNOT with the roles of the control and the target swapped

$$U_{SD}^{CNOT} = |+\rangle\langle +|_D \otimes \mathbb{I}^S + |-\rangle\langle -|_D \otimes \tilde{\sigma}_x^S,$$

where $\tilde{\sigma}_i$ correspond to Pauli matrices defined in the new basis.

(c) Use the result obtained in (b) to rewrite the interaction Hamiltonian (1) in the complementary bases.

(d) Hence deduce that (1) also governs the measurement, in which D acts as a measured system and S as a measurement device. Explain. What will the observable of D measured in this way be?

END OF PAPER