UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 1:30 pm to 4:30 pm

PAPER 65

QUANTUM INFORMATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Define the trace distance $D(\rho, \sigma)$ between two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ and prove that it can be expressed in the form:

$$D(\rho, \sigma) = \frac{1}{2}\left(\mathrm{Tr}Q + \mathrm{Tr}R\right), \tag{1}$$

where Q and R are suitably defined positive semi-definite operators in $\mathcal{B}(\mathcal{H})$.

(ii) Using the above identity, prove that

$$D(\rho, \sigma) = \max_{P} \mathrm{Tr}\left(P(\rho - \sigma)\right), \tag{2}$$

where the maximisation is over all projection operators $P \in \mathcal{B}(\mathcal{H})$.

(iii) Further, prove that

$$D(\rho, \sigma) = \max_{T} \mathrm{Tr}\left(T(\rho - \sigma)\right), \tag{3}$$

where the maximisation is over all positive semi-definite operators $T \in \mathcal{B}(\mathcal{H})$ with eigenvalues less than or equal to unity.

(iv) Let ρ be a quantum state and Λ be a linear completely positive trace-preserving map. Prove that

$$F_e(\rho, \Lambda) \leqslant \left(F(\rho, \Lambda(\rho))\right)^2, \tag{4}$$

where $F_e(\rho, \Lambda)$ denotes the entanglement fidelity, and $F(\rho, \Lambda(\rho))$ denotes the fidelity of the states ρ and $\Lambda(\rho)$.

2

The state of any qubit can be expressed in the form

$$\rho = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma}), \tag{1}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, with σ_x, σ_y, and σ_z being the Pauli matrices, and $\vec{s} = (s_x, s_y, s_z)$ is the Bloch vector.

(i) If $\vec{s} = (1/2, 1/3, 1/5)$, what is the probability that a projective measurement of the spin of the qubit along the X-axis will yield a value $+1$?

(ii) Find how the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ of a qubit changes under the action of an amplitude damping channel whose Kraus operators are given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.$$

(iii) What is the Bloch vector of the qubit after it is subjected to asymptotically many successive actions of the amplitude damping channel? Explain this by referring to the physical effect that this channel models.

(iv) Find an expression for the entanglement fidelity $F_e(\rho, \Lambda)$ if ρ is the state of a qubit and Λ is the amplitude damping channel.

3

(i) Let $p = \{p(x)\}_{x \in J}$ and $q = \{q(x)\}_{x \in J}$ be two probability distributions, with J being a finite alphabet. Define their relative entropy $D(p\|q)$ and prove that

$$D(p\|q) \geqslant 0. \tag{1}$$

(ii) For two states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, prove that the quantum relative entropy satisfies the inequality

$$S(\rho||\sigma) \geqslant 0. \tag{2}$$

(iii) Consider a bipartite system AB in an initial state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Let $\rho_{A'B'} := (\mathrm{id} \otimes \Lambda)\, \rho_{AB}$, where $\Lambda : \mathcal{D}(\mathcal{H}_B) \to \mathcal{D}(\mathcal{H}_B)$ denotes a quantum operation. Prove that

$$I(A' : B') \leqslant I(A : B), \tag{3}$$

where the notation $I(A : B)$ has been used to denote the quantum mutual information of the bipartite state σ_{AB}.

(iv) Consider the states $\rho = |0\rangle\langle 0|$ and $\sigma = (1 - \varepsilon/2)|0\rangle\langle 0| + \varepsilon/2|1\rangle\langle 1|$. Find a bound of the absolute value of the difference of their von Neumann entropies, in terms of ϵ.

4

(i) State the Holevo-Schumacher-Westmoreland Theorem.

1. Use it to obtain the product-state classical capacity of a qubit depolarizing channel Λ defined as follows:

$$\Lambda(\rho) = p\rho + \frac{1-p}{3}(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z), \tag{1}$$

   where σ_x, σ_y and σ_z are the Pauli matrices.

2. Consider an ensemble of quantum states $\mathcal{E} = \{p_x, \rho_x\}$ and let $\chi(\mathcal{E})$ denote its Holevo quantity. Let Λ be a quantum channel. Prove that

$$\chi(\mathcal{E}') \leqslant \chi(\mathcal{E}), \tag{2}$$

   where $\mathcal{E}' = \{p_x, \Lambda(\rho_x)\}$.

(ii) Consider a memoryless, quantum information source characterized by $\{\pi, \mathcal{H}\}$, where $\pi \in \mathcal{D}(\mathcal{H})$. Suppose on n uses, the source emits a signal state $|\Psi_k^{(n)}\rangle \in \mathcal{H}^{\otimes n}$ with probability $p_k^{(n)}$, the index k labelling the different possible signals. State the Typical Subspace Theorem, and use it to prove that for such a source there exists a reliable compression-decompression scheme of rate $R > S(\pi)$, where $S(\pi)$ denotes the von Neumann entropy of the source.

5

(i) The Bell states $|\Phi^+_{AB}\rangle$, $|\Phi^-_{AB}\rangle$, $|\Psi^+_{AB}\rangle$, and $|\Psi^-_{AB}\rangle$, can be characterized by two classical bits, namely, the parity bit and the phase bit. Show that the latter are eigenvalues of two commuting observables.

(ii) The Bell states form an orthonormal basis of the two-qubit Hilbert space. It is referred to as the Bell basis. Let us denote it by B_1. A sequence of two operations can be used to convert states of the computational basis $B_2 := \{|ij\rangle; i,j \in \{0,1\}\}$ to the Bell states. State what these operations are. Can they also be used to convert states of B_1 to B_2? Justify your answer.

(iii) Prove that the Schmidt rank of a pure state *cannot* be increased by local operations and classical communication (LOCC), clearly stating any theorem that you use.

(iv) It is known that a matrix A is doubly stochastic if and only if $\underline{x} \prec \underline{y}$, for all vectors $\underline{y}$, where $\underline{x} = A\underline{y}$.

Let $\rho \in \mathcal{D}(\mathcal{H})$ be a state, where $\dim \mathcal{H} = d$, and let $\Lambda : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ be a *unital* channel. Let $\underline{r} = (r_1, r_2, \ldots, r_d)$ and $\underline{s} = (s_1, s_2, \ldots, s_d)$ respectively denote the vectors of eigenvalues of ρ and $\sigma = \Lambda(\rho)$, arranged in non-increasing order. Using the above result, prove that $\underline{r} \prec \underline{s}$.

END OF PAPER