UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 9:00 am to 12:00 pm

PAPER 63

STRUCTURE AND EVOLUTION OF STARS

*Attempt no more than **THREE** questions.*
*There are **FOUR** questions in total.*
The questions carry equal weight.

You may use the equations and results given below without proof.

The symbols used in these equations have the meanings that were given in lectures. Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dm}{dr} = 4\pi r^2 \rho\,, \qquad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}\,, \qquad \frac{dL_{\mathrm{r}}}{dr} = 4\pi r^2 \rho\epsilon\,.$$

In a radiative region

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_{\mathrm{r}}}{16\pi a c r^2 T^3}\,.$$

In a convective region

$$\frac{dT}{dr} = \frac{(\Gamma_2 - 1)T}{\Gamma_2 P}\,\frac{dP}{dr}\,.$$

The luminosity, radius and effective temperature are related by $L = 4\pi R^2 \sigma T_{\mathrm{e}}^4\,.$

The equation of state for an ideal gas and radiation is $P = \dfrac{\mathcal{R}\rho T}{\mu} + \dfrac{aT^4}{3}\,,$
with $1/\mu = 2X + 3Y/4 + Z/2\,.$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) At time t a pre-mainsequence star of mass M and radius R is fully convective within its photosphere. It is composed of a mixture of fully ionized hydrogen and helium that behaves like a perfect gas with mean molecular weight μ and for which radiation pressure can be neglected. Explain why the pressure is given by $P = KT^{5/2}$, where T is the temperature and $K(t)$ is constant throughout the star.

Show that the central density and pressure behave as

$$\rho_\mathrm{c} \propto \frac{M}{R^3} \qquad \text{and} \qquad P_\mathrm{c} \propto \frac{M^2}{R^4}$$

and deduce that

$$K(t) = K_0 \mu^{-5/2} M^{-1/2} R^{-3/2}.$$

A suitable surface boundary condition is $\kappa P = \frac{2}{3} g$, where $\kappa = \kappa_0 \rho T^4$, with $\kappa_0 = \text{const}$, is the opacity in the atmosphere and g is the surface gravity. Deduce that the effective temperature T_e obeys

$$T_\mathrm{e}^8 = \frac{2}{3} G \kappa_0^{-1} K_0^{-2} \Re \mu^4 M^2 R,$$

where $\Re$ is the gas constant.

The gravitational energy of a polytrope of index n is $\Omega = -3(5-n)^{-1} G M^2 R^{-1}$. Show that

$$R^{-7/2} - R_0^{-7/2} = \frac{98\pi\sigma}{3} \left(\frac{2\Re}{3\kappa_0 G} \right)^{\frac{1}{2}} K_0^{-1} \mu^2 M^{-1} (t - t_0),$$

where $R = R_0$ when $t = t_0$, and σ is the Stefan–Boltzmann constant.

Deduce that the central temperature rises as $T_\mathrm{c} \propto \mu^{11/7} M^{5/7} t^{2/7}$ when $t \gg t_0$.

b) In a Hertzsprung–Russell diagram sketch the paths followed by two such pre-mainsequence stars of the same mass and metallicity $Z = 0.02$ but one with hydrogen abundance $X_1 = 0.7$ and the other with hydrogen abundance $X_2 = 0.8$ and indicate the direction of evolution.

Discuss briefly what would cause the evolution to deviate from such tracks.

c) The equation of state of a partially degenerate gas can be approximated by

$$P = \frac{\rho \Re T}{\mu_\mathrm{i}} + K_\mathrm{nr} \rho^{5/3},$$

where μ_i is the mean molecular weight of the ions and K_nr is a constant. When the star continues to collapse homologously show that the central temperature reaches a maximum

$$T_\mathrm{max} \propto M^{4/3} \mu_\mathrm{i}.$$

What does this mean for very low-mass stars?

2

In a plane-parallel atmosphere of negligible mass and containing no sources of energy the optical depth τ is defined by $d\tau = -\kappa\rho\, dz$, where $\kappa(\rho, T)$ is the total opacity of material of density ρ at temperature T, z is the height in the atmosphere and $\tau \to 0$ at large z. The equation of radiative transfer can be written in the form

$$\cos\theta \frac{dI}{d\tau} = I - \frac{j}{\kappa}, \qquad (*)$$

where $I(\tau, \theta)$ is the intensity of radiation at optical depth τ at an angle θ to the z-axis and j is the effective emissivity given by

$$\frac{j}{\kappa} = \frac{\sigma T^4}{\pi},$$

where σ is the Stefan–Boltzmann constant. Integrate $(*)$ over a sphere and use the fact that the flux F in the z direction is independent of τ to deduce that the mean intensity

$$J = \frac{1}{4\pi} \int_{\text{sphere}} I(\tau, \theta)\, d\Omega = \frac{j}{\kappa}.$$

Show that the form

$$I(\tau, \theta) = A(\tau) + C(\tau)\cos\theta$$

satisfies the Eddington closure approximation

$$cP_{\mathrm{r}} = \frac{4}{3}\pi J$$

between radiation pressure P_{r}, the speed of light c and the mean intensity. Deduce that

$$C = \frac{3F}{4\pi}$$

and that $(*)$ is satisfied if

$$\frac{dA}{d\tau} = C.$$

Use the fact that there is no flux into the star at $\tau = 0$ to find $A(\tau)$ and use the definition $\mathrm{F} = \sigma T_{\mathrm{e}}^4$ of effective temperature T_{e} to deduce that

$$T^4 = \frac{3}{4} T_{\mathrm{e}}^4 \left(\tau + \frac{2}{3}\right)$$

and that $T_0 = 2^{-1/4} T_{\mathrm{e}}$, where $T = T_0$ when $\tau = 0$.

In the atmosphere of a red dwarf the opacity

$$\kappa = \kappa_0 P^{\alpha-1} T^{4-4\beta},$$

where $\alpha > 0$ and $\beta > 0$ are constants, and radiation pressure is negligible. From hydrostatic equilibrium show that the pressure P varies with temperature T according to

$$P^{\alpha} = \frac{2\alpha g}{3\beta\kappa_0 T_0^4}\left(T^{4\beta} - T_0^{4\beta}\right),$$

where g is the surface gravity of the star.

Hence deduce that an approximate surface boundary condition is

$$\frac{P\kappa}{g} = \frac{4\alpha}{3\beta}(1 - 2^{-\beta})$$

at a radius r where the stellar luminosity $L_r = 4\pi\sigma r^2 T^4$.

3

In a frame in which all the material is corotating with angular velocity $\boldsymbol{\Omega}$ show that the equation of hydrostatic equilibrium in a star can be written as

$$\nabla P = -\rho \nabla \phi,$$

where P is the pressure, ρ is the density and $\phi(\boldsymbol{r})$ is a combined gravitational and centrifugal potential which satisfies

$$\nabla^2 \phi = 4\pi G \rho - 2\Omega^2.$$

Show that P and ρ must be constant on equipotential surfaces. Deduce that $\nabla^2\phi$ is constant on equipotential surfaces. Is $|\nabla\phi|$ necessarily constant on equipotential surfaces?

Argue that, for a star of uniform composition, temperature T is also constant on equipotential surfaces.

The star is in radiative equilibrium with heat flux

$$\boldsymbol{F} = -\chi \nabla T = -\chi \frac{dT}{d\phi} \nabla \phi,$$

where χ is the conductivity which is related to the opacity $\kappa(\rho, T)$ by

$$\chi = \frac{4acT^3}{3\kappa\rho},$$

a is the radiation constant and c is the speed of light. Show that the effective temperature on the surface of the star

$$T_{\mathrm{e}} \propto g^{1/4},$$

where g is the magnitude of the effective gravitational acceleration and sketch the cross-section of a rapidly spinning star, indicating where it is hottest.

Why is it not in general possible for the energy balance to be given simply by

$$\nabla.\boldsymbol{F} = \rho\epsilon,$$

where $\epsilon(\rho, T)$ is the energy generation rate per unit mass?

Now suppose that there is a steady circulation velocity field $\boldsymbol{v}(\boldsymbol{r})$ so that the energy balance is given instead by

$$\rho T \frac{Ds}{Dt} = \rho \boldsymbol{v}.T\nabla s = \rho\epsilon - \nabla.\boldsymbol{F},$$

where $s(\rho, T)$ is the specific entropy. Use continuity and the thermodynamic relation

$$T\, ds = dh - \frac{1}{\rho} dP,$$

where $h(\rho, T)$ is the specific enthalpy, to show that

$$\int_S \boldsymbol{F}.\boldsymbol{dS} = \int_V \rho\epsilon \, dV,$$

where S is an equipotential surface enclosing a volume V.

Hence show that the radiative gradient is given by

$$\frac{d\log T}{d\log P} = \frac{3\kappa PL}{16\pi acGmT^4}\left(1 - \frac{\Omega^2 V}{2\pi GM}\right)^{-1},$$

where L is the rate of energy generation in V and m is the mass in V.

4

Write an essay on the role of stars in the origin of the elements. Include in your discussion (a) the various phases of nuclear burning in stars of 1, 5 and 32 $M_\odot$, (b) how nucleosynthetic products are returned to the interstellar medium, (c) how and where elements with atomic masses greater than that of iron might be formed and (d) a brief discussion of the importance of binary stars.

END OF PAPER