UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 1:30 pm to 4:30 pm

PAPER 62

ASTROPHYSICAL FLUID DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

$$\nabla^2 \Phi = 4\pi G \rho$$

You may assume that for any vectors $\mathbf{C}$ *and* $\mathbf{D}$

$$\nabla \times (\mathbf{C} \times \mathbf{D}) = -\mathbf{D} \nabla \cdot \mathbf{C} + \mathbf{C} \nabla \cdot \mathbf{D} - \mathbf{C} \cdot \nabla \mathbf{D} + \mathbf{D} \cdot \nabla \mathbf{C}$$

and the components of $\mathbf{u} \cdot \nabla \mathbf{u}$, *for* $\mathbf{u} = (u_R, u_\phi, u_z)$, *in cylindrical coordinates* (R, ϕ, z) *are:*

$$\left(u_R \frac{\partial u_R}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_R}{\partial \phi} + u_z \frac{\partial u_R}{\partial z} - \frac{u_\phi^2}{R}, \;\; \frac{u_R}{R} \frac{\partial (R u_\phi)}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z}, \;\; u_R \frac{\partial u_z}{\partial R} + \frac{u_\phi}{R} \frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} \right)$$

$$\nabla \cdot \mathbf{u} = \frac{1}{R} \frac{\partial (R u_R)}{\partial R} + \frac{1}{R} \frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z}$$

1

A magnetized gas moving under the ideal MHD equations is such that in a Cartesian coordinate system (x, y, z), $\mathbf{u} = (u_x, 0, 0)$, and $\mathbf{B} = (0, 0, B_z)$. The motion is such that ρ, p, u_x, and B_z are functions only of x and the gravitational field is negligible. Show that the governing equations can be written in the form

$$\frac{\partial \mathsf{U}}{\partial t} + \mathsf{A}\frac{\partial \mathsf{U}}{\partial x} = \mathsf{0}\,,$$

where

$$\mathsf{U} = [\rho, p, u_x, B_z]^{\mathrm{T}}$$

is four dimensional state vector and A is a 4×4 matrix given by

$$\mathsf{A} = \begin{bmatrix} u_x & 0 & \rho & 0 \\ 0 & u_x & \gamma p & 0 \\ 0 & \frac{1}{\rho} & u_x & \frac{B_z}{\mu_0 \rho} \\ 0 & 0 & B_z & u_x \end{bmatrix}.$$

Show that A has a repeated eigenvalue $\lambda = u_x$ together with a pair of eigenvalues given by

$$\lambda_{\pm} = u_x \pm \sqrt{\left(\frac{{B_z}^2}{\mu_0 \rho} + \frac{\gamma p}{\rho}\right)}\,.$$

Relate these eigenvalues to the propagation of small amplitude perturbations propagating on a background for which U is constant and for which the space and time dependence is $\propto \exp(i(kx - \omega t))$, with ω and k being constant.

Show further that the governing equations have simple wave solutions corresponding to each of $\lambda_{\pm}$ which evolve according to the equations

$$\frac{\partial \lambda_{\pm}}{\partial t} + \lambda_{\pm}\frac{\partial \lambda_{\pm}}{\partial x} = 0\,.$$

2

Write down the equations governing the steady, spherically symmetric accretion of a barotropic gas for which $p = p(\rho)$ in a general spherically symmetric gravitational potential Φ. Show that the radial velocity, u_r, satisfies the equation

$$\frac{\left(u_r^2 - c_s^2\right)}{u_r}\frac{du_r}{dr} = \frac{2c_s^2}{r} - \frac{d\Phi}{dr}\,,$$

where

$$c_s^2 = \frac{dp}{d\rho}\,.$$

Explain what is meant by the statement that this equation has a critical point and write down, giving an explanation, the conditions that are required to be satisfied there. Show also that

$$\frac{1}{2}u_r^2 + h + \Phi = B\,,$$

where B is a constant, with

$$h(\rho) = \int_{\rho_1}^{\rho} \frac{c_s^2(\rho')}{\rho'} d\rho'\,,$$

where ρ_1 is an arbitrary constant density, is constant.

Spherically symmetric accretion onto a black hole is modeled by adopting the Paczynski-Wiita potential given by

$$\Phi = -\frac{GM}{r - r_G}\,,$$

where M is the mass, $r_G = 2GM/c^2$ is the Schwarzschild radius, and c is the speed of light. The gas has an isothermal equation of state, such that $p = \rho c_s^2$, where c_s is constant, and is uniform and at rest at infinity with density ρ_0.

Show that there is always a critical point with radius $r_{crit} > r_G$ and give an expression for r_{crit}. Show further that the accretion rate is given by

$$\dot{M} = \frac{\pi\rho_0(GM)^2}{4c_s^3}\left(1 + \beta/2 + \sqrt{1+\beta}\right)^2 \exp\left(\frac{4}{1+\sqrt{1+\beta}} - \frac{1}{2}\right)\,,$$

where $\beta = 16c_s^2/c^2$.

3

A steady state axisymmetric magnetohydrodynamic wind is such that the magnetic field may be written in the form

$$\mathbf{B} = (B_R, B_\phi, B_z) = -\frac{1}{R}\mathbf{e}_\phi \times \nabla\psi + B_\phi \mathbf{e}_\phi \,,$$

where ψ is the magnetic flux function and $\mathbf{e}_\phi$ is the unit vector in the azimuthal direction for cylindrical coordinates (R, ϕ, z). Show that the continuity equation is satisfied when the velocity is given by

$$\mathbf{u} = \frac{k\mathbf{B}}{\rho} + \mathbf{e}_\phi v(R, z) \,,$$

where $v(R, z)$ is an arbitrary function of R and z and $k(\psi)$ is an arbitrary function of ψ. Show that the steady state induction equation then becomes

$$\mathbf{B} \cdot \nabla \left(\frac{v(R, z)}{R} \right) = 0 \,,$$

and hence that $v(R, z)/R = \omega(\psi)$ is an arbitrary function of ψ alone.

Assuming that the gas pressure is negligible, show further that the steady state equation of motion implies that

$$\frac{k\mathbf{B}}{\rho} \cdot \nabla \left(\frac{1}{2}|\mathbf{u}|^2 + \Phi \right) = \frac{R\omega}{\mu_0 \rho}\mathbf{e}_\phi \cdot ((\nabla \times \mathbf{B}) \times \mathbf{B}) \,, \qquad \text{and}$$

$$\frac{k\mathbf{B}}{\rho} \cdot \nabla (Ru_\phi) = \frac{R}{\mu_0 \rho}\mathbf{e}_\phi \cdot ((\nabla \times \mathbf{B}) \times \mathbf{B}) \,.$$

Hence show that

$$\frac{1}{2}(u_R^2 + u_z^2 + (u_\phi - R\omega)^2) + \Phi - \frac{1}{2}R^2\omega^2 = \epsilon(\psi) \,,$$

where $\epsilon(\psi)$ depends only on ψ. Assuming the wind is launched near $z = 0$ where B_ϕ is negligible, describe qualitatively the motion in that vicinity.

The poloidal magnetic field lines in the vicinity of the plane $z = 0$ are straight lines inclined at an angle α to the z axis. The footpoint of the field line labeled by ψ, located at $R = R_0, z = 0$, rotates with angular velocity, $\omega(\psi)$, given by

$$\omega^2(\psi) = \frac{1}{R}\left.\frac{\partial \Phi}{\partial R}\right|_{R=R_0, z=0} \,.$$

Show that when $\Phi \propto 1/\sqrt{R^2 + z^2}$, the condition for the wind to accelerate freely away from the vicinity of $z = 0$ is that $\alpha > \pi/6$.

4

A non magnetic barotropic star, for which $p = p(\rho)$, rotates uniformly about the z axis with angular velocity Ω so that in cylindrical coordinates (R, ϕ, z), it has a steady axisymmetric structure with $\mathbf{u} = \mathbf{u}_0 = (0, R\Omega, 0)$.

Show that

$$H = h(\rho) + \Phi - \frac{1}{2}R^2\Omega^2$$

is constant, where

$$h(\rho) = \int_{\rho_1}^{\rho} \frac{c_s^2(\rho')}{\rho'} d\rho' ,$$

with ρ_1 being an arbitrary constant density and the square of the sound speed $c_s^2(\rho) = dp/d\rho$. The star is subject to small amplitude perturbations so that the velocity becomes $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, where the velocity perturbation $\mathbf{u}'$ is of the form

$$\mathbf{u}' = (v_R(R, z), v_\phi(R, z), v_z(R, z)) \ \exp(i\omega t \ + \ im\phi) ,$$

with $m > 0$. The gravitational potential perturbation may be neglected. Show that the components of the linearized equations of motion yield

$$i\sigma v_R - 2\Omega v_\phi = -\frac{\partial W}{\partial R} ,$$

$$i\sigma v_\phi + 2\Omega v_R = -\frac{imW}{R} ,$$

$$i\sigma v_z = -\frac{\partial W}{\partial z} ,$$

where $\sigma = \omega + m\Omega$ and $p'/\rho = c_s^2\rho'/\rho = W(R, z)\exp(i\omega t + im\phi)$, with p', and ρ' being the pressure and density perturbations respectively.

Show also that linearization of the continuity equation yields

$$i\sigma\frac{W\rho}{c_s^2} = -\frac{1}{R}\frac{\partial(R\rho v_R)}{\partial R} - \frac{im\rho v_\phi}{R} - \frac{\partial(\rho v_z)}{\partial z} .$$

Show that the linearized equations can be reduced to a single equation for W of the form

$$(4\Omega^2 - \sigma^2)\frac{W\rho}{c_s^2} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\rho\frac{\partial W}{\partial R}\right) - \frac{m^2\rho W}{R^2} + \left(1 - \frac{4\Omega^2}{\sigma^2}\right)\frac{\partial}{\partial z}\left(\rho\frac{\partial W}{\partial z}\right) + \frac{2m\Omega}{R\sigma}\frac{\partial \rho}{\partial R}W .$$

For slow rotation and in the low frequency anelastic approximation the term on the left hand side is neglected. In addition it can be assumed that ρ becomes a function of $\sqrt{R^2 + z^2}$ alone so that

$$\frac{1}{R}\frac{\partial \rho}{\partial R} = \frac{1}{z}\frac{\partial \rho}{\partial z} .$$

Show that in this limit a solution for W of the form $W = zR^m$ exists if

$$\sigma = \frac{2\Omega}{m + 1} .$$

END OF PAPER