MATHEMATICAL TRIPOS  Part III

Friday, 1 June, 2012  1:30 pm to 3:30 pm

PAPER 61

DYNAMICS OF ASTROPHYSICAL DISCS

Attempt no more than TWO questions.

There are THREE questions in total.

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.
The vertical structure of a thin Keplerian accretion disc containing a mixture of gas and radiation is governed by the equations

\[ \frac{\partial p}{\partial z} = -\rho \Omega^2 z, \quad (1) \]

\[ p = p_g + p_r = \frac{k \rho T}{\mu_m m_p} + \frac{4 \sigma T^4}{3c}, \quad (2) \]

\[ \frac{\partial F}{\partial z} = \frac{9}{4} \mu \Omega^2, \quad (3) \]

\[ F = -\frac{16 \sigma T^3}{3 \kappa \rho} \frac{\partial T}{\partial z}. \quad (4) \]

(i) Explain briefly the physical interpretation of these equations.

(ii) Assuming that the ratio \( \lambda = p_r/p_g \) is non-zero and independent of \( z \), and that \( \mu_m \) is a constant, show that the solution of equations (1) and (2) is of the form

\[ T = T_0 \left(1 - \frac{z^2}{H^2}\right)^3, \quad \rho = \rho_0 \left(1 - \frac{z^2}{H^2}\right)^3, \quad p = p_0 \left(1 - \frac{z^2}{H^2}\right)^4 \]

for \( |z| < H \), where \( H, T_0, \rho_0 \) and \( p_0 \) are independent of \( z \) (but may depend on the radial coordinate \( r \)). Express \( T_0, \rho_0 \) and \( p_0 \) in terms of \( H \) and \( \lambda \) (as well as \( \Omega \) and the physical constants appearing in the equations). Show also that the surface density \( \Sigma \) satisfies

\[ 420 \lambda (\lambda + 1)^3 \Sigma = \left(\frac{\mu_m m_p k}{\kappa}\right)^4 \frac{\sigma}{c} H^7 \Omega^6. \]

(iii) Assuming further that \( \kappa \) is constant, show that

\[ \mu = \left(\frac{\lambda}{\lambda + 1}\right) \frac{4c}{9 \kappa} \]

for \( |z| < H \).

(iv) Although in this model the vertical profile of the effective viscosity \( \mu \) differs from that of the pressure, a Shakura–Sunyaev alpha parameter can be defined in a vertically averaged way by

\[ \nu \Sigma = \int_{-H}^{H} \mu \, dz = \frac{\alpha}{\Omega} \int_{-H}^{H} p \, dz, \]

where \( \nu \) is the mean effective kinematic viscosity and \( \Sigma \) is the surface density. Show that

\[ \frac{\lambda}{\lambda + 1} = \frac{\alpha H \Omega}{8c} \kappa \Sigma \]

and

\[ \nu = \frac{\alpha H^2 \Omega}{9}. \]
Assuming that $\alpha$ is independent of $r$, deduce that the disc has a mean effective kinematic viscosity $\bar{\nu}(r, \Sigma)$ with the limiting dependences

$$
\bar{\nu} \propto \begin{cases} 
    r \Sigma^{2/3}, & \lambda \ll 1, \\
    r^{3/2} \Sigma^{-2}, & \lambda \gg 1.
\end{cases}
$$

Comment briefly on the viscous stability of the disc in the limits in which it is dominated by gas pressure or radiation pressure.

[You may assume that

$$
\int_0^1 (1 - x^2)^n \, dx = \frac{2^{2n} (n!)^2}{(2n + 1)!} = \left( \frac{2n}{2n + 1} \right) \left( \frac{2n - 2}{2n - 1} \right) \cdots \frac{2}{3}
$$

for positive integers $n$.]

\[ \text{Part III, Paper 61} \]
In the local approximation for astrophysical discs, a steady elliptical vortex patch occupies the region
\[ \frac{x^2}{b^2} + \frac{y^2}{a^2} < 1, \]
where \( a > b > 0 \). The total velocity field inside the vortex has the form \( \mathbf{u} = \alpha y \mathbf{e}_x - \beta x \mathbf{e}_y \), where \( \alpha \) and \( \beta \) are constants. The velocity field at large distance from the vortex is \( \mathbf{u} = -Sx \mathbf{e}_y \). The relative vorticity \( (\nabla \times \mathbf{u})_z \) is equal to \(-S\) outside the vortex and \(-S + \zeta_0\) inside the vortex. The fluid may be considered to be incompressible and inviscid. You may assume without proof that
\[ \frac{\zeta_0}{S} = -\frac{(r + 1)}{r(r - 1)}, \]
where \( r = a/b \).

(i) Determine the constants \( \alpha \) and \( \beta \) in terms of \( r \) and \( S \).

(ii) Show that the pressure \( p \) inside the vortex is given by \( p/\rho = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 + \text{constant} \), and determine the constants \( A \) and \( B \) in terms of \( r \), \( S \), and \( \Omega \) (the angular velocity of the frame of reference).

(iii) Formulate the equation of motion of a particle (in two dimensions) inside the vortex, if the particle is subject to a drag force \(-\gamma(\dot{x} - u)\) per unit mass, where \( \gamma \) is a constant drag coefficient, \( \mathbf{x}(t) \) is the position vector of the particle and \( \mathbf{u}(\mathbf{x}) \) is the velocity of the fluid. Show that there exist solutions of the form \( \mathbf{x} \propto e^{\lambda t} \), where \( \lambda \) is a root of the quartic equation
\[ \lambda^4 + 2\gamma \lambda^3 + [2\Omega(2\Omega - S) + \gamma^2] \lambda^2 + 2\Omega(\alpha + \beta - S) \gamma \lambda + \alpha \beta \gamma^2 = 0. \]

(iv) You may assume that the solutions of the real quartic equation \( \lambda^4 + c_3 \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0 \) all have \( \text{Re}(\lambda) < 0 \) if and only if \( c_3, c_2, c_1 \) and \( c_0 \) are all positive and \( c_3 c_2 c_1 > c_3^2 c_0 + c_1^2 \). Show that particles of all drag coefficients inside the vortex are attracted to its centre if and only if
\[ 2\Omega(2\Omega - S) > \Omega(\alpha + \beta - S) > \alpha \beta > 0. \]
Show also that these conditions are satisfied in a Keplerian disc if \( r > 3 \).
In the local approximation, a two-dimensional (‘razor-thin’) inviscid isothermal disc satisfies the equation of mass conservation,
\[ \frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{u}) = 0, \]
and the equation of motion,
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2 \mathbf{\Omega} \times \mathbf{u} = -\nabla \Phi - \nabla \Phi_{d,m} - \frac{1}{\Sigma} \nabla \left( c_s^2 \Sigma \right), \]
where \( \Sigma(x, y, t) \) is the surface density, \( \mathbf{u}(x, y, t) \) is the two-dimensional velocity field, \( \mathbf{\Omega} = \mathbf{\Omega} e_z \) is the angular velocity of the frame of reference, and \( \Phi = -\Omega Sx^2 \) is the effective potential, where \( S \) is the rate of orbital shear. The gravitational potential of the disc, \( \Phi_{d}(x, y, z, t) \), satisfies
\[ \nabla^2 \Phi_{d} = 4\pi G \Sigma \delta(z), \]
and \( \Phi_{d,m}(x, y, t) = \Phi_{d}(x, y, 0, t) \) is its value in the midplane. The isothermal sound speed \( c_s \) is a constant.

(i) Formulate the linearized equations for infinitesimal disturbances from a basic state in which \( \Sigma \) is a constant and \( \mathbf{u} = -Sx \mathbf{e}_y \).

(ii) Show that solutions exist in which the velocity perturbation \( \mathbf{v} \) has the form of a shearing wave,
\[ \mathbf{v}(x, y, t) = \text{Re} \left\{ \tilde{\mathbf{v}}(t) \exp[i \mathbf{k}(t) \cdot \mathbf{x}] \right\}, \]
and similarly for the other perturbations, provided that
\[ \frac{d\mathbf{k}}{dt} = Sk_y \mathbf{e}_x. \]
Show that the shearing-wave amplitudes evolve according to the ordinary differential equations
\[ \frac{d\tilde{\Sigma}'}{dt} + \Sigma (ik_x \tilde{v}_x + ik_y \tilde{v}_y) = 0, \]
\[ \frac{d\tilde{v}_x}{dt} - 2\Omega \tilde{v}_y = -ik_x \left( c_s^2 - \frac{2\pi G \Sigma}{k} \right) \frac{\tilde{\Sigma}'}{\Sigma}, \]
\[ \frac{d\tilde{v}_y}{dt} + (2\Omega - S) \tilde{v}_x = -ik_y \left( c_s^2 - \frac{2\pi G \Sigma}{k} \right) \frac{\tilde{\Sigma}'}{\Sigma}, \]
where \( k = (k_x^2 + k_y^2)^{1/2} \), and \( k_x \) depends on \( t \) as given above.

(iii) Show that the quantity
\[ q' = \frac{ik_x \tilde{v}_y - ik_y \tilde{v}_x}{\Sigma} - \frac{(2\Omega - S)\tilde{\Sigma}'}{\Sigma^2} \]
is independent of time, and describe briefly the physical interpretation of this quantity.
(iv) For perturbations such that $k_y = 0$, show that

\[
\left( \frac{d^2}{dt^2} + c_s^2 k_x^2 - 2\pi G\Sigma |k_x| + \kappa^2 \right) \dot{\Sigma}' = -2\Omega \Sigma^2 \dot{q}',
\]

where $\kappa^2 = 2\Omega(2\Omega - S)$. Assuming that $\kappa^2 > 0$, describe how the behaviour of the solutions depends on the value of the Toomre parameter

\[
Q = \frac{c_s \kappa}{\pi G\Sigma}.
\]