UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 1:30 pm to 3:30 pm

PAPER 61

DYNAMICS OF ASTROPHYSICAL DISCS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The vertical structure of a thin Keplerian accretion disc containing a mixture of gas and radiation is governed by the equations

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z\,, \tag{1}$$

$$p = p_{\mathrm{g}} + p_{\mathrm{r}} = \frac{k\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}} + \frac{4\sigma T^4}{3c}\,, \tag{2}$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\mu\Omega^2\,, \tag{3}$$

$$F = -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z}\,. \tag{4}$$

(i) Explain briefly the physical interpretation of these equations.

(ii) Assuming that the ratio $\lambda = p_{\mathrm{r}}/p_{\mathrm{g}}$ is non-zero and independent of z, and that μ_{m} is a constant, show that the solution of equations (1) and (2) is of the form

$$T = T_0\left(1 - \frac{z^2}{H^2}\right)\,, \quad \rho = \rho_0\left(1 - \frac{z^2}{H^2}\right)^3\,, \quad p = p_0\left(1 - \frac{z^2}{H^2}\right)^4$$

for $|z| < H$, where H, T_0, ρ_0 and p_0 are independent of z (but may depend on the radial coordinate r). Express T_0, ρ_0 and p_0 in terms of H and λ (as well as Ω and the physical constants appearing in the equations). Show also that the surface density Σ satisfies

$$420\lambda(\lambda+1)^3\Sigma = \left(\frac{\mu_{\mathrm{m}} m_{\mathrm{p}}}{k}\right)^4 \frac{\sigma}{c} H^7\Omega^6\,.$$

(iii) Assuming further that κ is constant, show that

$$\mu = \left(\frac{\lambda}{\lambda+1}\right)\frac{4c}{9\kappa}$$

for $|z| < H$.

(iv) Although in this model the vertical profile of the effective viscosity μ differs from that of the pressure, a Shakura–Sunyaev alpha parameter can be defined in a vertically averaged way by

$$\bar{\nu}\Sigma = \int_{-H}^{H} \mu\,\mathrm{d}z = \frac{\alpha}{\Omega}\int_{-H}^{H} p\,\mathrm{d}z\,,$$

where $\bar{\nu}$ is the mean effective kinematic viscosity and Σ is the surface density. Show that

$$\frac{\lambda}{\lambda+1} = \frac{\alpha H\Omega}{8c}\kappa\Sigma$$

and

$$\bar{\nu} = \frac{\alpha H^2\Omega}{9}\,.$$

Assuming that α is independent of r, deduce that the disc has a mean effective kinematic viscosity $\bar{\nu}(r, \Sigma)$ with the limiting dependences

$$\bar{\nu} \propto \begin{cases} r\Sigma^{2/3}\,, & \lambda \ll 1\,, \\ r^{3/2}\Sigma^{-2}\,, & \lambda \gg 1\,. \end{cases}$$

Comment briefly on the viscous stability of the disc in the limits in which it is dominated by gas pressure or radiation pressure.

[You may assume that

$$\int_0^1 (1 - x^2)^n \, \mathrm{d}x = \frac{2^{2n}(n!)^2}{(2n+1)!} = \left(\frac{2n}{2n+1}\right)\left(\frac{2n-2}{2n-1}\right)\cdots\frac{2}{3}$$

for positive integers n.]

2

In the local approximation for astrophysical discs, a steady elliptical vortex patch occupies the region

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} < 1\,,$$

where $a > b > 0$. The total velocity field inside the vortex has the form $\boldsymbol{u} = \alpha y\, \boldsymbol{e}_x - \beta x\, \boldsymbol{e}_y$, where α and β are constants. The velocity field at large distance from the vortex is $\boldsymbol{u} = -Sx\, \boldsymbol{e}_y$. The relative vorticity $(\boldsymbol{\nabla} \times \boldsymbol{u})_z$ is equal to $-S$ outside the vortex and $-S + \zeta_0$ inside the vortex. The fluid may be considered to be incompressible and inviscid. You may assume without proof that

$$\frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)}\,,$$

where $r = a/b$.

(i) Determine the constants α and β in terms of r and S.

(ii) Show that the pressure p inside the vortex is given by $p/\rho = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 + \text{constant}$, and determine the constants A and B in terms of r, S and Ω (the angular velocity of the frame of reference).

(iii) Formulate the equation of motion of a particle (in two dimensions) inside the vortex, if the particle is subject to a drag force $-\gamma(\dot{\boldsymbol{x}} - \boldsymbol{u})$ per unit mass, where γ is a constant drag coefficient, $\boldsymbol{x}(t)$ is the position vector of the particle and $\boldsymbol{u}(\boldsymbol{x})$ is the velocity of the fluid. Show that there exist solutions of the form $\boldsymbol{x} \propto \mathrm{e}^{\lambda t}$, where λ is a root of the quartic equation

$$\lambda^4 + 2\gamma\lambda^3 + [2\Omega(2\Omega - S) + \gamma^2]\lambda^2 + 2\Omega(\alpha + \beta - S)\gamma\lambda + \alpha\beta\gamma^2 = 0\,.$$

(iv) You may assume that the solutions of the real quartic equation $\lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 = 0$ all have $\mathrm{Re}(\lambda) < 0$ if and only if c_3, c_2, c_1 and c_0 are all positive and $c_3c_2c_1 > c_3^2c_0 + c_1^2$. Show that particles of all drag coefficients inside the vortex are attracted to its centre if and only if

$$2\Omega(2\Omega - S) > \Omega(\alpha + \beta - S) > \alpha\beta > 0\,.$$

Show also that these conditions are satisfied in a Keplerian disc if $r > 3$.

3

In the local approximation, a two-dimensional ('razor-thin') inviscid isothermal disc satisfies the equation of mass conservation,

$$\frac{\partial \Sigma}{\partial t} + \boldsymbol{\nabla} \cdot (\Sigma \boldsymbol{u}) = 0\,,$$

and the equation of motion,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}\Phi_{\mathrm{d,m}} - \frac{1}{\Sigma}\boldsymbol{\nabla}(c_{\mathrm{s}}^2 \Sigma)\,,$$

where $\Sigma(x,y,t)$ is the surface density, $\boldsymbol{u}(x,y,t)$ is the two-dimensional velocity field, $\boldsymbol{\Omega} = \Omega\,\boldsymbol{e}_z$ is the angular velocity of the frame of reference, and $\Phi = -\Omega S x^2$ is the effective potential, where S is the rate of orbital shear. The gravitational potential of the disc, $\Phi_{\mathrm{d}}(x,y,z,t)$, satisfies

$$\nabla^2 \Phi_{\mathrm{d}} = 4\pi G \Sigma\, \delta(z)\,,$$

and $\Phi_{\mathrm{d,m}}(x,y,t) = \Phi_{\mathrm{d}}(x,y,0,t)$ is its value in the midplane. The isothermal sound speed c_{s} is a constant.

(i) Formulate the linearized equations for infinitesimal disturbances from a basic state in which Σ is a constant and $\boldsymbol{u} = -Sx\,\boldsymbol{e}_y$.

(ii) Show that solutions exist in which the velocity perturbation $\boldsymbol{v}$ has the form of a shearing wave,

$$\boldsymbol{v}(x,y,t) = \mathrm{Re}\left\{\tilde{\boldsymbol{v}}(t)\exp[\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}]\right\}\,,$$

and similarly for the other perturbations, provided that

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = S k_y\,\boldsymbol{e}_x\,.$$

Show that the shearing-wave amplitudes evolve according to the ordinary differential equations

$$\frac{\mathrm{d}\tilde{\Sigma}'}{\mathrm{d}t} + \Sigma(\mathrm{i}k_x \tilde{v}_x + \mathrm{i}k_y \tilde{v}_y) = 0\,,$$

$$\frac{\mathrm{d}\tilde{v}_x}{\mathrm{d}t} - 2\Omega \tilde{v}_y = -\mathrm{i}k_x\left(c_{\mathrm{s}}^2 - \frac{2\pi G\Sigma}{k}\right)\frac{\tilde{\Sigma}'}{\Sigma}\,,$$

$$\frac{\mathrm{d}\tilde{v}_y}{\mathrm{d}t} + (2\Omega - S)\tilde{v}_x = -\mathrm{i}k_y\left(c_{\mathrm{s}}^2 - \frac{2\pi G\Sigma}{k}\right)\frac{\tilde{\Sigma}'}{\Sigma}\,,$$

where $k = (k_x^2 + k_y^2)^{1/2}$, and k_x depends on t as given above.

(iii) Show that the quantity

$$\tilde{q}' = \frac{\mathrm{i}k_x\tilde{v}_y - \mathrm{i}k_y\tilde{v}_x}{\Sigma} - \frac{(2\Omega - S)\tilde{\Sigma}'}{\Sigma^2}$$

is independent of time, and describe briefly the physical interpretation of this quantity.

(iv) For perturbations such that $k_y = 0$, show that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c_\mathrm{s}^2 k_x^2 - 2\pi G\Sigma|k_x| + \kappa^2\right)\tilde{\Sigma}' = -2\Omega\Sigma^2\tilde{q}' \,,$$

where $\kappa^2 = 2\Omega(2\Omega - S)$. Assuming that $\kappa^2 > 0$, describe how the behaviour of the solutions depends on the value of the Toomre parameter

$$Q = \frac{c_\mathrm{s}\kappa}{\pi G\Sigma} \,.$$

END OF PAPER