UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 1:30 pm to 4:30 pm

PAPER 60

ORIGIN AND EVOLUTION OF GALAXIES

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

i) Sketch the evolution of the radius of the outer mass shell of an overdense region in an expanding Universe of collisionless matter. Explain why the mass shell virializes at half the turnaround radius.

ii) Assume the Universe to be Einstein-de-Sitter and matter dominated with Hubble constant $H(t)$ and background density $\bar{\rho} = 3H^2(t)/8\pi G$. Show that for an overdense homogeneous sphere of collisonless matter with radius $R(t)$ the solution,

$$\tau = \frac{1}{\sqrt{\Delta_{\rm ta}}}\left[\frac{1}{2}\arcsin(2y-1) - \sqrt{y-y^2} + \frac{\pi}{4}\right]$$

with

$$\Delta_{\rm ta} = \left(\frac{3\pi}{4}\right)^2, \quad \tau = H(t_{\rm ta})\, t \quad \text{and} \quad y = R(t)/R(t_{\rm ta}),$$

satisfies the equation of motion $\ddot{R} = -GM/R^2$, where M is the mass interior to R.

Use the approximation

$$\tau \approx \frac{9}{3\pi}\, y^{3/2}\left[1 + \frac{3y}{10}\right] \qquad \text{for} \quad y << 1,$$

to show that for the linearly extrapolated density contrast $\delta = (\rho - \bar{\rho})/\bar{\rho}$ at collapse when $R(t_{\rm coll} = 2\, t_{\rm ta}) = 0$,

$$\delta(t_{\rm coll}) = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3}.$$

2

A thin disc galaxy with radius R_d and surface mass density profile $\Sigma(R) = \Sigma_\mathrm{d}(R/R_\mathrm{d})^{-1}$ of the baryonic mass has been observed at redshift $z = 1$. The disc radius subtends an angle of 7/12 arcsec on the sky and the rotational velocity inferred from spectroscopy is $v_\mathrm{rot}^\mathrm{d} = 210$ km/s. Assume the Universe to be flat with matter density and cosmological constant parameters, $\Omega_\mathrm{mat} = 1 - \lambda = 0.25$, baryonic density parameter $\Omega_\mathrm{bar} = 0.05$ and assume the Hubble constant to evolve with redshift as $H(z) = 70\,(0.25\,(1+z)^3 + 0.75)^{1/2}$ km/s Mpc^{-1}.

i) Calculate the linear size of the disc radius R_d from its angular extent.

ii) Assume that after virialization of the halo hosting the disc galaxy all baryons have settled into centrifugal support conserving angular momentum with the specific angular momentum of the baryons at the edge of the disc $j_\mathrm{d}(R_\mathrm{d})$ equal to the specific angular momentum of the halo at the virial radius $j_\mathrm{h}(r_\mathrm{vir})$. Assume further that the ratio of rotational velocity at the virial radius to virial velocity of the halo, $v_\mathrm{rot}^\mathrm{h}(r_\mathrm{vir}) = 0.1\,v_\mathrm{vir}$, and that the dark matter does not contribute to the gravitational potential at $R \leqslant R_\mathrm{d}$. At what redshift is the galaxy expected to have formed? State and explain any additional assumptions you make.

[You may find the following approximation useful: $\int_0^1 (0.25\,(1+z)^3 + 0.75)^{-1/2}\,\mathrm{d}z \approx 0.8$.]

3

The volume cooling rate of gas with temperature T can be written as $\mathcal{C}(\mathcal{T}) = \Lambda(T)\, n_{\mathrm{H}}^2$, where $\Lambda(T)$ is the cooling function and n_{H} is the number density of hydrogen.

i) Sketch a diagram of the cooling function of metal-free primordial gas for $10^4\,\mathrm{K} < T < 10^8\,\mathrm{K}$. Discuss the main characteristic features of the cooling curve and the processes responsible. How does the curve change with the metallicity of the gas? What happens if the gas becomes photoionized? What processes are available for cooling at $T < 10^4\,\mathrm{K}$?

ii) Sketch the curve in the number density-temperature plane, at which the cooling time of a uniform cloud of metal-free primordial gas with temperature equal to the virial temperature equals its free-fall time. Discuss briefly how this diagram has been used to explain why galaxies have characteristic masses in the range $10^8 - 10^{12} M_\odot$. You may want to include lines of constant mass and lines of constant virial density at characteristic redshifts in the diagram to support your discussion. Pay particular attention to explaining how the cooling diagram has been used to explain the upper mass limit of galaxies.

iii) Discuss briefly how the shape of the mass function of dark matter haloes and the luminosity function of galaxies are related, discussing the physical processes believed to shape the galaxy luminosity function. You may find it helpful to sketch diagrams of the mass/luminosity function and the efficiency of star formation as a function of dark matter halo mass to support your discussion.

[You may find the following helpful for labelling your diagrams. For metal-free primordial gas: $\log(\Lambda\,(T = 15000\,\mathrm{K})/(\mathrm{erg\,cm^3})) \approx -21.8$, $\log(\Lambda\,(T = 10^7\,\mathrm{K})/(\mathrm{erg\,cm^3})) \approx -23.0$. The number density of hydrogen at redshift $z = 0$ is $n_{\mathrm{H}} \approx 2 \times 10^{-7}\mathrm{cm^{-3}}$. The Boltzmann constant can be approximated as $k_{\mathrm{B}} \approx 1.38 \times 10^{-16}\,\mathrm{erg\,K^{-1}}$ and the Gravitational constant as $G \approx 6.67 \times 10^{-8}\,\mathrm{cm^3\,s^{-2}\,g^{-1}}$.]

4

i) With the Press-Schechter ansatz, the mass fraction of the matter density in the Universe in collapsed objects with mass greater than M is,

$$f(>M,t) = \mathrm{erfc}\left(\frac{\delta_c(t)}{\sqrt{2}\sigma(M)}\right) \quad \text{with} \quad \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp\left(-t^2\right) \mathrm{d}t.$$

Explain the meaning of $\delta_\mathrm{c}(t)$ and $\sigma(M)$. Use the Press-Schechter ansatz to derive a mass function of collapsed objects $n(M,t)$.

ii) The mass function of collapsed objects has the approximate form

$$n(M,t) \approx A\, M^\alpha \exp\left[-(M/M_*(t))^\beta\right]$$

with suitable constants A, α and β. The characteristic mass at redshift $z=3$ and $z=7$ has been inferred from observations to be $M_*(z=3) \approx 10^{13} M_\odot$ and $M_*(z=7) \approx 6.25\times 10^{11} M_\odot$. Approximate the rms fluctuation amplitude of the matter density smoothed with a top-hat window function with comoving radius R and the corresponding power spectrum as a power-law, $\sigma(R) \propto R^m$ and $P(k) \propto k^n$, and calculate m and n. Compare your value of n with the canonical value of $n=1$ for primordial density fluctuations in the early Universe and explain briefly why they are different.

iii) It has been inferred from observations that $\sigma(R = 7h^{-1}\mathrm{Mpc}, z = 3) \approx 0.25$ where h is the Hubble constant in units of 100 km/s Mpc^{-1}. Estimate at which redshift the characteristic number density $M\, \mathrm{d}n/\mathrm{d}M$ of objects with mass $10^{10} M_\odot$ becomes $10^{-4}\mathrm{Mpc}^{-3}$.

[Assume that the mean matter density at $z=0$ is $1.41\times 10^{11} M_\odot \mathrm{Mpc}^{-3}$ and that $h=0.7$.]

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