UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 5 June, 2012 9:00 am to 12:00 pm

PAPER 6

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let Ω be a domain in $\mathbb{R}^n$.

(i) Define what it means for $v \in L^1_{\text{loc}}(\Omega)$ to be the α-th weak derivative of $u \in L^1_{\text{loc}}(\Omega)$.

(ii) Let $u, v \in L^1_{\text{loc}}(\Omega)$. Show that v is the α-th weak derivative of u if and only if there exists a sequence of smooth functions u_j on Ω such that $u_j \to u$ in $L^1(\Omega')$ and $D^\alpha u_j \to v$ in $L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$.

(iii) Let $u, v \in W^{1,1}_{\text{loc}}(\Omega)$. Show that $u + v \in W^{1,1}_{\text{loc}}(\Omega)$ with weak derivative $Du + Dv$.

(iv) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function with $\sup_{\mathbb{R}} |f'| < \infty$ and let $u \in W^{1,1}_{\text{loc}}(\Omega)$. Show that $f(u) \in W^{1,1}_{\text{loc}}(\Omega)$ with weak derivative $f'(u)Du$.

2

Let Ω be a bounded domain in $\mathbb{R}^n$. Consider the uniformly elliptic operator

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du$$

for $u \in W^{1,2}(\Omega)$ and a^{ij}, b^i, c^j, and d bounded measurable functions on Ω.

(i) Define what it means for $u \in W^{1,2}(\Omega)$ to be a weak solution to the Dirichlet problem

$$Lu = D_i f^i + g \text{ in } \Omega, \qquad u = \varphi \text{ on } \partial\Omega$$

for $f, g \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$.

(ii) Suppose $\int_\Omega (-b^i D_i \zeta + d\zeta) \leqslant 0$ for all $\zeta \in W^{1,1}_0(\Omega)$ with $\zeta \geqslant 0$. State and prove the weak maximum principle for a weak solution $u \in W^{1,2}(\Omega)$ to $Lu = 0$ in Ω.

(iii) Show that the Dirichlet problem in Part (i) has at most one solution provided $\int_\Omega (-b^i D_i \zeta + d\zeta) \leqslant 0$ for all $\zeta \in W^{1,1}_0(\Omega)$ with $\zeta \geqslant 0$.

(iv) Show by an explicit example that the Dirichlet problem in Part (i) can have more than one solution when the condition $\int_\Omega (-b^i D_i \zeta + d\zeta) \leqslant 0$ fails for some $\zeta \in W^{1,1}_0(\Omega)$ with $\zeta \geqslant 0$.

3

Let Ω be a domain in $\mathbb{R}^n$. Consider the uniformly elliptic operator

$$Lu = a^{ij} D_{ij} u + b^i D_i u + cu$$

for $u \in C^2(\Omega)$ and $a^{ij}, b^i, c \in C^{0,\mu}(\Omega)$ for some $\mu \in (0,1)$.

(i) Let $f \in C^{0,\mu}(\Omega)$ and suppose that $u \in C^2(\Omega)$ is a solution to $Lu = f$ in Ω. Show that $u \in C^{2,\mu}(\Omega)$.

(ii) Now let $a^{ij}, b^i, c, f \in C^{1,\mu}(\Omega)$ and suppose that $u \in C^2(\Omega)$ is a solution to $Lu = f$ in Ω. Using Part (i), show that $u \in C^{3,\mu}(\Omega)$.

(iii) Let a^{ij}, b^i, c, and f be smooth functions on Ω. Use Part (ii) to show that if $u \in C^2(\Omega)$ is a solution to $Lu = f$ in Ω, then u is smooth on Ω.

[You may use without proof any standard result on existence of solutions.]

4

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\mu \in (0,1)$. Let $a^{ij}, b^i, c, f \in C^{0,\mu}(\overline{\Omega})$ with $c \leqslant 0$. Consider the elliptic operator $Lu = a^{ij}D_{ij}u + b^i D_i u + cu$ for $u \in C^2(\Omega)$.

(i) Define what it means for $v \in C^0(\Omega)$ to be a subsolution to $Lu = f$ in Ω.

(ii) Suppose $v \in C^0(\overline{\Omega})$ is a subsolution and $w \in C^0(\overline{\Omega})$ is a supersolution to $Lu = f$ in Ω. Show that if $v \leqslant w$ on $\partial\Omega$ then $v \leqslant w$ in Ω.

(iii) Suppose $v \in C^0(\overline{\Omega})$ is a subsolution to $Lu = f$ in Ω and let $B_R(x_0) \subset\subset \Omega$. Define $V \in C^0(\overline{\Omega})$ such that $V = v$ on $\overline{\Omega} \setminus B_R(x_0)$ and V is the solution to $LV = f$ in $B_R(x_0)$ with $V = v$ on $\partial B_R(x_0)$. Show that V is a subsolution to $Lu = f$ in Ω.

(iv) Let $\varphi \in C^0(\partial\Omega)$. For $x \in \Omega$, let

$$U(x) = \sup\{v(x) : v \in C^0(\overline{\Omega}) \text{ is a subsolution to } Lu = f \text{ in } \Omega \text{ with } v \leqslant \varphi \text{ on } \partial\Omega\}.$$

Show that the function U is well-defined and is a C^2 solution to $LU = f$ in Ω.

[You may use assume the following three facts:

(a) Given any open ball $B_R(x_0) \subset\subset \Omega$ and $\psi \in C^0(\partial B_R(x_0))$, there exists a unique solution $u \in C^0(\overline{B_R(x_0)}) \cap C^{2,\mu}(B_R(x_0))$ to $Lu = f$ in $B_R(x_0)$ and $u = \psi$ on $\partial B_R(x_0)$.

(b) If $v \in C^0(\overline{\Omega})$ is a subsolution to $Lu = f$ in Ω, then

$$\sup_{\Omega} v \leqslant \sup_{\partial\Omega} v^+ + C \sup_{\Omega} |f|$$

where $v^+(x) = \max\{v(x), 0\}$ and $C > 0$ is a constant depending only on n, Ω, and L.

(c) If $v, w \in C^0(\overline{\Omega})$ are subsolutions to $Lu = f$ in Ω, then $\max\{v, w\}$ is also a subsolution.]

5

Let Ω be a domain in $\mathbb{R}^n$.

(i) State and prove the mean value property for C^2 harmonic functions in Ω.

(ii) State and prove Weyl's lemma.

(iii) State and prove the Harnack inequality for C^2 harmonic functions in Ω.

6

Let Ω be a domain in $\mathbb{R}^n$.

(i) State without proof the weak Harnack inequality for a weak supersolution $u \in W^{1,2}(\Omega)$ to

$$D_i(a^{ij}D_iu + b^iu) + c^jD_ju + du = D_if^i + g \text{ in } \Omega,$$

where $f^i \in L^q(\Omega)$, $g \in L^{q/2}(\Omega)$ for some $q > n$ and a^{ij}, b^j, c^j, d are bounded measurable functions on Ω with

$$a^{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2 \text{ for some } \lambda > 0 \text{ and all } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

(ii) State and prove the strong maximum principle for a weak subsolution $u \in W^{1,2}(\Omega)$ to

$$D_i(a^{ij}D_iu + b^iu) + c^jD_ju + du = 0 \text{ in } \Omega,$$

where a^{ij}, b^j, c^j, d are bounded measurable functions on Ω with

$$a^{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2 \text{ for some } \lambda > 0 \text{ and all } x \in \Omega,\ \xi \in \mathbb{R}^n.$$

(iii) Let $f^i \in L^2(B_1(0)) \cap L^q(B_1(0))$, $g \in L^2(B_1(0)) \cap L^{q/2}(B_1(0))$ for some $q > n$ and let $\varphi \in C^0(\partial B_1(0))$. Show that there exists a unique solution $u \in C^0(\overline{B_1(0)}) \cap W^{1,2}_{\text{loc}}(B_1(0))$ to

$$\Delta u = D_if^i + g \text{ weakly in } B_1(0), \qquad u = \varphi \text{ pointwise on } \partial B_1(0)$$

[You may assume there exists a solution $u \in W^{1,2}(B_1(0))$ to $\Delta u = D_if^i + g$ in $B_1(0)$ with $u - \varphi \in W^{1,2}_0(B_1(0))$ whenever $\varphi \in W^{1,2}(B_1(0))$.]

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