UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 7 June, 2012 9:00 am to 12:00 pm

PAPER 59

GALACTIC ASTRONOMY AND DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A galaxy has the potential

$$\Phi(R, z) = \frac{v_0^2}{2} \log(R^2 + z^2 q^{-2}),$$

where v_0 and q are constants and (R, ϕ, z) are cylindrical polar coordinates. What is the density distribution that generates this potential?

For which values of the parameters q and v_0 is the density everywhere positive definite?

Derive the velocity of circular orbits in the equatorial plane ($z = 0$) as a function of radius R. Hence, plot the rotation curve of the galaxy.

Explain why we may seek a distribution function $F = F(E, L_z^2)$, where E is the energy and L_z the angular momentum of the stellar orbits.

Using angled brackets to denote averages over the distribution function, demonstrate that the velocity dispersion tensor has the following properties

$$\langle v_R^2 \rangle = \langle v_z^2 \rangle ,$$

and

$$\langle v_R v_z \rangle = 0 = \langle v_R v_\phi \rangle = \langle v_\phi v_z \rangle .$$

Verify that the distribution function has the form

$$F(E, L_z^2) = A L_z^2 \exp(-4E/v_0^2) + B \exp(-2E/v_0^2),$$

where A and B are constants to be determined.

[*Hint: You are reminded of the standard integral* ($\alpha > 0$)

$$\int_{-\infty}^{\infty} \exp(-\alpha v^2) dv = \sqrt{\frac{\pi}{\alpha}} . \quad]$$

2

Derive the virial theorem for a galaxy in the form

$$2T_{ij} + \Pi_{ij} + W_{ij} = 0\,,$$

where T_{ij}, Π_{ij} and W_{ij} are the kinetic energy, pressure and potential energy tensors respectively.

Suppose the galaxy has a spherically symmetric potential, $\Phi = \Phi(r)$, where r is the spherical polar radius. Consider a pressure-supported (that is, $T_{ij} = 0$) tracer population of stars with a flattened density $\rho(R, z)$, where (R, z) are cylindrical polar coordinates. Show that the virial ratio

$$\frac{\Pi_{RR}}{\Pi_{zz}} = \frac{\displaystyle\int \rho(R,z)\Phi'(r)\frac{R^2}{r}d^3r}{\displaystyle\int \rho(R,z)\Phi'(r)\frac{z^2}{r}d^3r}\,,$$

where a prime denotes differentiation.

If the tracer density has the form

$$\rho(R, z) = \rho_0(R^2 + z^2q^{-2})^{-\gamma/2}\,,$$

where ρ_0, q and γ are constants, show that the virial ratio is well-defined and has the value

$$\frac{\Pi_{RR}}{\Pi_{zz}} = \frac{\displaystyle\int_0^\pi \sin^3\theta(\cos^2\theta + q^{-2}\sin^2\theta)^{-\gamma/2}d\theta}{\displaystyle\int_0^\pi \sin\theta\cos^2\theta(\cos^2\theta + q^{-2}\sin^2\theta)^{-\gamma/2}d\theta}\,.$$

In the limit of modest flattening ($q \to 1$), show that

$$\frac{\Pi_{RR}}{2\Pi_{zz}} = 1 + \frac{\gamma}{5}(q^{-2} - 1)\,.$$

Interpret this result physically.

[*Hint: you are reminded of the standard integral* ($m > 0$)

$$\int_0^\pi \sin^m\theta d\theta = \frac{\sqrt{\pi}\,\Gamma(1/2 + m/2)}{\Gamma(1 + m/2)}$$

where $\Gamma(x)$ *denotes the Gamma function.*]

3

By separation of variables in cylindrical polar coordinates (R, ϕ, z), show that Laplace's equation has solutions

$$\Phi(R, z) = \exp(\pm kz) J_0(kR)$$

where k is a constant and J_0 is the Bessel function of index zero.

By using Gauss' theorem, show that the potential of an infinitesimally thin disk of surface density $\Sigma(R)$ confined to the plane $z = 0$ is given by

$$\Phi(R, z) = \int_0^\infty S(k) J_0(kR) \exp(-k|z|) dk \,,$$

where

$$S(k) = -2\pi G \int_0^\infty J_0(kR) \Sigma(R) R dR \,.$$

Show that the velocity of circular orbits in the disk plane is given by

$$v_{\mathrm{c}}^2(R) = -R \int_0^\infty S(k) J_1(kR) k dk \,,$$

where J_1 is the Bessel function of index unity.

If the surface density in the disk behaves like

$$\Sigma(R) = \frac{\Sigma_0 R_0}{R} \,,$$

where Σ_0 and R_0 are constants, show that

$$v_{\mathrm{c}}^2(R) = GM(R)/R \,,$$

where $M(R)$ is the mass interior to radius R.

Is this result surprising? Explain your answer.

[*Hint: you may assume the solution to the equation*

$$\frac{1}{u} \frac{d}{du} \left(u \frac{dJ}{du} \right) + J = 0$$

finite at $u = 0$ *is the Bessel function* $J_0(u)$*. You may also assume Hankel's formula*

$$F(r) = \int_0^\infty k dk \int_0^\infty R dR \, F(R) J_0(kR) J_0(kr) \,,$$

as well as the following properties of Bessel functions

$$\int_0^\infty J_0(x) dx = 1, \qquad \int_0^\infty J_1(x) dx = 1, \qquad \frac{dJ_0(x)}{dx} = -J_1(x) \,. \quad]$$

4

Suppose there are two galaxies G_1 and G_2 which are to first approximation point-like with masses M_1 and M_2. G_1 and G_2 are in circular motion about their common centre of mass O. If the separation of G_1 and G_2 is R, show that the angular velocity of the line G_1G_2 in the centre of mass frame is

$$|\underline{\omega}|^2 = \frac{G(M_1+M_2)}{R^3}\,.$$

Let us consider the equilibrium points of a small dwarf galaxy G with mass m, which is much smaller than M_1 and M_2. Let us introduce Cartesian (x, y, z) with origin at O, and with z-axis parallel to $\underline{\omega}$ and x-axis along the line G_1G_2. Give a physical reason why the equilibrium points lie in the plane $z = 0$.

Show that the equilibrium points are the stationary points of

$$E(x,y) = -\frac{m\omega^2}{2}|\underline{r}_{\mathrm{G}}|^2 - \frac{GmM_1}{|\underline{r}_{\mathrm{G}} - \underline{r}_1|} - \frac{GmM_2}{|\underline{r}_{\mathrm{G}} - \underline{r}_2|}$$

where $\underline{r}_{\mathrm{G}}$, $\underline{r}_1$ and $\underline{r}_2$ are the position vectors of G, G_1 and G_2 respectively.

Let $\underline{r}_{\mathrm{G}} = (x, y, 0)$. Using R as the unit of length and $G(M_1+M_2)/R^2$ as the unit of energy, show that the stationary points on the line joining G_1 and G_2 are the extrema of

$$F(x) = -\frac{x^2}{2} - \frac{1-\alpha}{|x+\alpha|} - \frac{\alpha}{|x+\alpha-1|}$$

where $\alpha = M_2/(M_1+M_2)$.

Solve this equation in the limit of small α, and show that there are three equilibrium points:

$$\begin{aligned} \mathrm{L}_1 &= \left(1-(\alpha/3)^{1/3}, 0, 0\right), \\ \mathrm{L}_2 &= \left(1+(\alpha/3)^{1/3}, 0, 0\right), \\ \mathrm{L}_3 &= \left(-1-5\alpha/12, 0, 0\right), \end{aligned}$$

.

Now, by introducing plane polars with origin at G_1, show that there are two further equilibrium points L_4 and L_5 such that triangles $G_1G_2\,L_4$ and $G_1G_2\,L_5$ are equilateral.

Sketch the surface $E(x, y)$ and mark the points $L_1, \ldots L_5$.

END OF PAPER