UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 1 June, 2012 1:30 pm to 4:30 pm

PAPER 58

BLACK HOLES

*Attempt **ALL** of Section I*
*and **TWO** of the **THREE** questions from Section II.*

Section II carries twice the weight of Section I.
The three questions within Section II carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

(i) The motion of a point particle of non-zero mass m may be deduced from the action

$$S = \int d\lambda \left\{ \frac{dx^\mu}{d\lambda} p_\mu - \frac{1}{2} e \left(p^2 + m^2 \right) \right\} .$$

By eliminating the variables $e(x)$ and $p_\mu(x)$, or otherwise, show that this action is equivalent to the action $S = -m \int d\tau$, where τ is the particle's proper time.

(ii) Let the vector field ℓ be normal to a null hypersurface $\mathcal{N}$. Why is ℓ also tangent to $\mathcal{N}$? Show that ℓ is tangent to a null geodesic in $\mathcal{N}$ (i.e. to a "generator" of $\mathcal{N}$).

(iii) What is the "future domain of dependence" $D^+(\Sigma)$ of a partial Cauchy surface Σ of a spacetime $\mathcal{M}$? Sketch the CP diagram for the Reissner-Nordstrom (RN) metric, and use it to briefly explain the meaning of the term "future Cauchy horizon". Also explain briefly why it is believed that the Cauchy horizon of the RN metric would become singular if the back-reaction of any "test" particle that approaches it could be properly taken into account.

(iv) What is a null geodesic congruence? What is the significance of the expansion θ of the congruence? State, with brief justification, a bound on the value of θ for generators of the event horizon. Explain briefly the distinction betwen the event horizon and an "apparent horizon", illustrating your answer with a Finkelstein diagram.

(v) Give reasons, either mathematical or physical, why no local definition of energy exists for non-stationary spacetimes whereas no such difficulty arises for charge. Write down the ADM formula for the total energy of an asymptotically-flat spacetime. State the "dominant energy condition" on the matter stress tensor $T_{\mu\nu}$ that is needed in the proof that the ADM energy is non-negative.

(vi) A given spacetime metric admits a Killing vector field ξ. Write down the Komar surface integral for the associated charge $Q_\xi(V)$ contained within a volume V on a spacelike hypersurface Σ (you may ignore the normalization factor). Show how this can be rewritten as a volume integral of the form $\int_V dS_\mu J^\mu_\xi$. Using Einstein's field equations, express J^μ_ξ in terms of the stress tensor $T_{\mu\nu}$ and the vector field ξ, and show that $D_\mu J^\mu_\xi = 0$.

SECTION II

1

(a) Define the term "Killing horizon". Given that $\mathcal{N}$ is a non-degenerate Killing horizon of the Killling vector field $\xi = \partial/\partial\alpha$, explain how its surface gravity κ arises from the failure of α to affinely parametrize the generators of $\mathcal{N}$. Hence, or otherwise, show that

$$\partial_\mu \, \xi^2\big|_{\mathcal{N}} = -2\kappa \, \xi_\mu\big|_{\mathcal{N}} \, .$$

(b) A static spacetime has the metric

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \, , \qquad (*)$$

where

$$F(r) = 1 - r^2/R^2 \, .$$

Compute the magnitude a of the 4-acceleration of an observer at $r < R$ on an orbit of the Killing vector field $k = \partial/\partial t$ [i.e., at fixed (r, θ, φ)]. Use your result to show that t is proper time for *inertial* static observers.

(c) By introducing Eddington-Finkelstein-type coordinates, ingoing (v, r, θ, φ) or outgoing (u, r, θ, φ), show that the apparent singularity of the metric $(*)$ at $r = R$ is a coordinate singularity. Show further that the hypersurface $r = R$ is a Killing horizon of the Killing vector field k, and find its surface gravity κ.

(d) Using the relation between surface gravity and the Hawking temperature, which you should first state, deduce the temperature, in thermal equilibrium, experienced by a static inertial observer in the spacetime with metric $(*)$. State the Tolman law for the local temperature of a static spacetime in thermal equilibrium, and use it to find the local temperature $T(r)$. How does $T(r)$ compare to the acceleration a of a static observer as $r \to R$? By considering the form of the metric $(*)$ near $r = R$, explain why your result is expected from the Unruh effect for an observer at constant proper acceleration in 2D Minkowski spacetime.

(e) Rewrite the metric $(*)$ in the coordinates (t, χ, θ, ϕ), where the new angular radial coordinate χ is defined by

$$r = R \sin\chi \, .$$

Explain why an apparent singularity of the metric at $\chi = \pi/2$ is only a coordinate singularity. Use your results to deduce the CP diagram for its maximally-analytic extension.

2

(a) A stationary axisymmetric black hole spacetime in coordinates (t, r, θ, φ) has Killing vector fields $k = \partial/\partial t$ and $m = \partial/\partial\varphi$. Given that its event horizon is a Killing horizon of the Killing vector field $\xi = k + \Omega m$, for some constant Ω, show that, on the event horizon,

$$k^2 = \Omega^2 m^2 \, .$$

State, with justification, the physical interpretation of the constant Ω.

(b) The Kerr metric in Boyer-Lindquist coordinates is

$$\begin{aligned} ds^2 \;&=\; -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta\, d\varphi^2 \\ &\quad - 4Ma\,\frac{r\sin^2\theta}{\Sigma}\, dt d\varphi \;+\; \frac{\Sigma}{\Delta} dr^2 \;+ \Sigma d\theta^2 \, , \end{aligned}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Explain briefly the nature of the curvature singularity of this metric, and why the radial coordinate r may be negative. State the relation of the parameter a to the angular momentum J and explain why the singularity is a naked one if $J^2 > M^4$, i.e. for an over-rotating Kerr black hole. Is there an event horizon in this case?

(c) By considering how m^2 behaves near the singularity of the Kerr metric, explain how an over-rotating Kerr black hole would allow the construction of a time machine.

(d) Define the term "ergoregion" for a stationary black hole spacetime. In the case of a Kerr black hole, does the outer boundary of this region, the "ergosphere", intersect the event horizon? Illustrate your answer with a sketch of the two surfaces.

(e) On the axis of symmetry in Kerr-Schild coordinates, the Kerr metric reduces to

$$ds^2 = -d\tilde{t}^2 + dr^2 + \frac{2Mr}{r^2 + a^2}\left(d\tilde{t} + dr\right)^2$$

Show that the radial coordinate for a unit-mass particle of energy ε falling into a Kerr black hole $(M > a)$ along the axis of symmetry obeys an equation of the form

$$\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) = \varepsilon^2 \, ,$$

where you should find the effective potential V_{eff}. Show that the particle reaches $r = 0$ only if $\varepsilon^2 \geqslant 1$. What happens subsequently to a particle with $\varepsilon^2 > 1 + M/a$? Illustrate your answer using a CP diagram.

3

Write an essay on the relation of black holes to thermodynamics.

You should start with a statement of the laws of black hole mechanics and explain why they are analogous to the laws of thermodynamics. In particular, by consideration of the first laws, you should explain how the connection between surface gravity and temperature leads to the Bekenstein–Hawking formula for the entropy of a black hole.

Next, you should consider a quantum scalar field Φ satisfying the wave equation $D^{\mu}\partial_{\mu}\Phi = 0$ in a globally hyperbolic non-stationary spacetime that is asymptotic to Minkowski spacetime in the far past and far future, and explain how the vacuum state can evolve to a non-vacuum state. You should then explain briefly how your results apply to late-time Hawking radiation from a Schwarzschild black hole formed from gravitational collapse.

You should conclude with a brief discussion of some of the implications of Hawking radiation for both black holes and thermodynamics.

END OF PAPER